STEP Support Programme

STEP 3 Matrices Topic Notes

These notes are designed to help support the study of the matrices topic, which has been introduced into the new STEP syllabus (first examination in 2019) in light of the changes to the A-level specifications.

Note that all of the content of the STEP 2 matrices topic (as well as the rest of the STEP 1 and STEP 2 content) is assumed knowledge for STEP 3.

The determinant of a 3×3 matrix

Just as the determinant of a 2×2 transformation matrix gives the area scale factor of the transformation, the determinant of a 3×3 transformation matrix gives the **volume scale factor** of the transformation. A negative determinant indicates that the orientation of the image is reversed by the transformation, as in a reflection, for example.

Calculating a determinant

The determinant of the 3×3 matrix $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is given by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Here we have taken the elements of the first row of $\mathbf{A}$ and multiplied each one by the determinant obtained by crossing out its row and column in $\mathbf{A}$. (For example, c is multiplied by the determinant obtained by crossing out the first row and third column of $\mathbf{A}$.) This smaller determinant is called a *minor* of $\mathbf{A}$. The plusses and minuses in the calculation are determined according to the position of the element in the following alternating pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

We can calculate the determinant by expanding along any row or column of $\mathbf{A}$ in this way. For example, if we expand using the second column, we would get

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

On occasions, one row or column is much easier to use than others. For example, if one row has two zeros, then expanding the determinant using that row would give us just a single 2×2 determinant to calculate. As an example:

$$\begin{vmatrix} 1 & 4 & 6 \\ 0 & 0 & 2 \\ 5 & -3 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 4 \\ 5 & -3 \end{vmatrix} = -2 \times (-23) = 46,$$

by expanding along the second row.

This method of calculating determinants also extends to calculating the determinant of an $n \times n$ matrix; in the pattern of plusses and minuses, there is always a $+$ in the top left-hand corner.

If we fully expand any of the above expressions for $\det \mathbf{A}$, we obtain

$$\det \mathbf{A} = aei + bfg + cdh - afh - bdi - ceg$$

Another way to think about this expression is that we multiply the elements in each of the six diagonals (where a diagonal is allowed to "wrap around" the edges of the matrix, so the diagonals include a-e-i, b-f-g and a-f-h); we add together the products of the forward diagonals (those which go down to the right) and subtract the products of the backward diagonals (those which go down to the left). This approach, though, does not generalise to higher-dimensional matrices.

$$\left|\begin{matrix} a & \searrow & b & \searrow & c \\ d & \searrow & e & \searrow & f \\ g & & h & & i \end{matrix}\right| \begin{matrix} \searrow \\ \searrow \\ \end{matrix} \qquad \begin{matrix} \swarrow \\ \swarrow \\ \end{matrix} \left|\begin{matrix} a & \swarrow & b & \swarrow & c \\ d & \swarrow & e & \swarrow & f \\ g & & h & & i \end{matrix}\right|$$

forward diagonals   backward diagonals

Simplifying determinants using row and column operations

Sometimes a determinant can be simplified using the following row and/or column operations. These can also be used to calculate any determinant, as we shall see. We will not attempt to prove that these operations are valid in general. In the 2×2 and 3×3 cases, they can be checked using the formula for calculating a determinant, but this is quite tedious and not necessarily enlightening.

- **Taking out a common factor.** If k is a common factor of all of the elements in a row or a column, it can be taken out as a factor of the determinant. For example:

$$\begin{vmatrix} 3 & 1 & -2 \\ 4 & 5 & 6 \\ 0 & -1 & -6 \end{vmatrix} = -2 \begin{vmatrix} 3 & 1 & 1 \\ 4 & 5 & -3 \\ 0 & -1 & 3 \end{vmatrix}$$

  Be careful if two or more columns or rows share a common factor; each column or row needs to be treated separately by this rule, for example:

$$\begin{vmatrix} 6 & -2 & 4 \\ 12 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 2 \begin{vmatrix} 3 & -1 & 2 \\ 12 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 4 \begin{vmatrix} 3 & -1 & 2 \\ 6 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

- **Adding or subtracting a multiple of a row/column to a different row/column.** We can add or subtract a multiple of one row to a different row. Likewise, we can add or subtract a multiple of one column to a different column. For example, in the determinant

$$\begin{vmatrix} 6 & -2 & 8 \\ 4 & 3 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$

  we notice that the first row is almost the same as twice the third row: two of the three elements in each row are the same. So we can subtract twice the third row from the first row to get

$$\begin{vmatrix} 6 & -2 & 8 \\ 4 & 3 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & -6 & 0 \\ 4 & 3 & 2 \\ 3 & 2 & 4 \end{vmatrix} \quad (\mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_3)$$

  which we can now easily evaluate by expanding along the first row. The comment for the reader, $(\mathbf{r}_1 \to \mathbf{r}_1 - 2\mathbf{r}_3)$, means that the first row has been replaced by the first row minus twice the third row. (This notation might also be useful in exams to explain your working to the examiner.)

A particularly useful application is this. If a determinant has two identical rows, then we can subtract one from the other. This will result in a row which is all zero. If we then expand the determinant along this row, we find that the determinant equals zero. Similarly, if one row is a multiple of another, we can use row operations to get a row which is all zero, and so the determinant equals zero in the same way as before. The same also applies to two columns which are identical or where one column is a multiple of another. We will see examples of how this can be effectively used in the collection of matrices questions.

Another result that follows from this is that if any two rows of a determinant are swapped, then the determinant is negated (i.e. the sign of the determinant changes). Similarly, if any two columns are swapped, then the determinant is also negated. (Can you prove this using row/column operations?)

It also turns out that by a judicious choice of row operations, we can transform any determinant into one which has at most one non-zero entry in some row, and is therefore very easy to calculate. When we then expand the determinant, we have to calculate a smaller determinant, and this can be done in the same way. In general, this is an efficient way to calculate determinants, and is far less effort than working with the adjugate matrix. Nevertheless, the adjugate is useful theoretically and for proving results about matrices.

The inverse of a 3×3 matrix

To find the inverse of a 3×3 matrix you can find the *adjugate matrix* and divide throughout by the determinant, as explained in "Method 1" of this wikiHow. An alternative approach is to use row operations instead, as explained in "Method 2" of the above wikiHow. For example, to use this method to find the inverse of $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$, start by adjoining the identity matrix to $\mathbf{A}$ to get:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Then perform row operations until the LHS of the above matrix is the 3×3 identity matrix, and then the RHS will be $\mathbf{A}^{-1}$.

This method has the advantage of being more efficient for larger matrices. Do beware, though, that if a question specifies a method to use, you must use it.

Solving simultaneous equations

Consider the simultaneous equations:

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

Each of these three equations is the equation of a plane in 3 dimensions. The equations can be written in the form $\mathbf{Ax} = \mathbf{d}$, where $\mathbf{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$. If $\mathbf{A}^{-1}$ exists (which it will as long as $\det \mathbf{A} \neq 0$) then there is a unique solution to the simultaneous equations given by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{d}$.

If, on the other hand, $\det \mathbf{A} = 0$ then the equations do not have unique solution. There are several different situations which could occur geometrically, and you need to look at the three equations more closely to work out which situation you have.[1]

To start with, consider whether any of the planes described by the equations are parallel or the same plane. This will be the case when 2 (or 3) rows of $\mathbf{A}$ are multiples of each other. In this case, the possibilities are:

- The three planes described by the equations are all identical. In this case, all three equations are the same (though they might be multiplied throughout by a constant), i.e., we have $(a_1, b_1, c_1, d_1) = k(a_2, b_2, c_2, d_2) = m(a_3, b_3, c_3, d_3)$ for some k and m. There are infinitely many solutions (all the points on this plane), and it takes 2 parameters to describe the general solution algebraically.
- The three planes described by the planes are all parallel, but they are not all the same plane (though two of them might be). In this case there will be no solutions. Here we have $(a_1, b_1, c_1) = k(a_2, b_2, c_2) = m(a_3, b_3, c_3)$ but $d_1 \neq kd_2$ or $d_1 \neq md_3$ or $kd_2 \neq md_3$.
- Two of the planes are the same (so we have $(a_i, b_i, c_i, d_i) = k(a_j, b_j, c_j, d_j)$ for some i and j), but the third plane is not parallel, so it will cut the other plane to create a line of solutions. It takes 1 parameter to describe the general solution algebraically.
- Two of the planes are parallel but not the same (and hence there are no solutions), and the third plane cuts across these two.

If none of the planes are parallel, but $\det \mathbf{A} = 0$, then there are two possible situations.

- The three planes meet in a line of solutions – this is sometimes called a "sheaf" of planes. If you eliminate one of the variables from two pairs of equations (for example, eliminate x from the first and second equations, and also eliminate x from the first and third equations) you will end up with two *consistent* equations (i.e., two equations which are just multiples of each other) which describe how y and z are related along this line; together with one of the equations, this will give an expression for the line of solutions involving 1 parameter.
- The three planes form the sides of an infinitely long triangular prism, and there are no solutions as there are no points which are on all three planes at the same time. When you try to eliminate a variable in this case you will end up with two *inconsistent* equations.

[1] In 2 dimensions things are much easier. Each equation represents a line, and either the lines cross (a unique solution exists), the lines are parallel (no solutions) or the lines are the same line (infinitely many solutions).