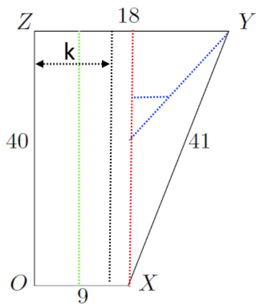
STEP Support Programme

STEP II Mechanics Questions: Solutions

These are not fully worked solutions — you need to fill in some gaps. It is a good idea to look at the "Hints" document before this one.

1 **(i)** Divide the lamina into a rectangle and a triangle. Let the Centre of Mass lie on a line a distance k from OZ.



The CoM of the triangle is a horizontal distance 3cm from the red line, so a distance $12 - k$ from the CoM.
The CoM of the rectangle is a distance of $k - 4.5$ from the CoM.
The triangle has twice the area so twice the mass - say the rectangle has mass $2M$ and the triangle has mass M. *Alternatively, you could use a mass density ρ instead of a mass*
Taking moments, $2Mg(k - 4.5) = Mg(12 - k)$ which gives $k = 7$ as required.

(ii) First, work out the areas of each part of the tank:
Front $41d$, Back $40d$, Base $9d$, each side 540.
As the sheet metal has constant density, we can choose units such that the mass per unit area is 1, and use these values as the mass in our moments calculations.
Let the centre of mass be a distance q from the back of the tank.
Taking moments about the centre of mass:
$40dq + 9d(q - 4.5) = 540.2.(7 - q) + 41d(13.5 - q)$ (back, base, 2 sides and front).
This solves to give

$$q = \frac{3(140 + 11d)}{5(12 + d)} \text{ cm}$$

as required.

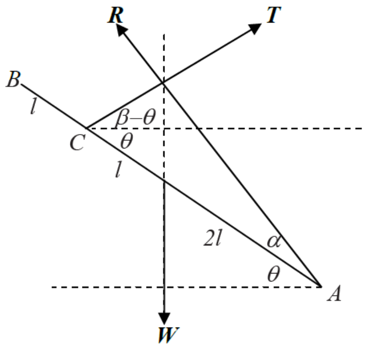
When $d = 20$, $q = \frac{27}{4}$, the area of the sheet metal is 2880 and the volume of the tank is 10800. The ratio of the mass of the metal to the mass of the liquid is thus $2880 : 10800k$ which simplifies to $4 : 15k$.
From part (i), the mass of the liquid will act at a point 7cm from the back of the tank, and from part (ii) and given $d = 20$, the mass of the metal will act at a point $\frac{27}{4}$ from the back of the tank.

Let p be the distance of the Centre of Mass of the filled tank from the back of the tank. Taking moments, $4(p - \frac{27}{4} = 15k(7-p)$ which simplifies to give

$$p = \frac{3(35k+9)}{4+15k}$$

At this point, it's a good idea to do a quick sanity check - what happens when the liquid is very much heavier than the tank? (i.e. k is very large.) p is then approximately 7, which is what we would expect as the CoM of the liquid is a distance of 7 from the back of the tank.

2 **(i)** Start, as with virtually all mechanics questions, by drawing a nice clear diagram showing the rod and the forces acting:



As the rod is held in equilibrium, we know it neither slips nor rotates. We can take moments, and resolve forces horizontally and vertically. Taking moments,

$$2lW\cos\theta = 3lT\sin\beta$$

so

$$T = \frac{2W\cos\theta}{3\sin\beta}$$

Resolving horizontally:

$$R\cos(\alpha+\theta) = T\cos(\beta-\theta)$$

Substituting in for T:

$$R = \frac{2W\cos\theta\cos(\beta-\theta)}{3\sin\beta\cos(\alpha+\theta)}$$

Resolving vertically:

$$R\sin(\alpha+\theta) + T\sin(\beta-\theta) = W)$$

Substituting for R and T:

$$\frac{2W\cos\theta\cos(\beta-\theta)\sin(\alpha+\theta)}{3\sin\beta\cos(\alpha+\theta)} + \frac{2W\cos\theta\sin(\beta-\theta)}{3\sin\beta} = W$$

Cancelling the Ws, using the relevant formulae to expand the trig functions, and taking EXTREME CARE with your $\sin, \cos, \alpha, \beta$ and θ terms leads to the required answer.

(ii) Given that $\theta = 30°$ and $\beta = 45°$,

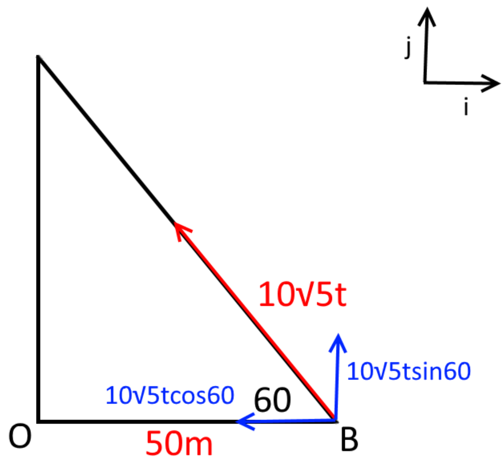
$$\cot\alpha = 3\tan 30 + 2\cot 45 = \sqrt{3} + 2$$

In order to show that $\alpha = 15°$, it is sufficient to show that $\tan 15° = \frac{1}{\sqrt{3}+2}$, which can be done by considering $\tan(60° - 45°)$

3 **(i)** As $\theta = \arctan\frac{1}{2}$, $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$.
Vertical distance travelled at time t:

$$s = 25\sin\theta t - \frac{1}{2}gt^2$$
$$= \frac{25t}{\sqrt{5}} - 5t^2$$

Horizontally, $s = 25\cos\theta t = 10\sqrt{5}t$, in a direction 60° clockwise of the East-West line BO.



We are taking the origin to be at O, so the initial position of the particle is $50\mathbf{i}$. Thus, the position vector relative to O at time t is

$$(50 - 5\sqrt{5}t)\mathbf{i} + 5\sqrt{15}t\mathbf{j} + (5t(\sqrt{5} - t))\mathbf{k}$$

Taking out a factor of 5 and then finding the magnitude gives:

$$5\sqrt{100 - 20\sqrt{5}t + 5t^2 + 15t^2 + 5t^2 - 2\sqrt{5}t^3 + t^4}$$

and we can square the given expression to show the required result.
The distance is shortest when $t^2 - \sqrt{5}t + 10$ is at a minimum. Completing the square gives $t = \frac{\sqrt{5}}{2}$ at P.
Substituting this value of t into our formula for the position vector gives

$$OP = \frac{75}{2}\mathbf{i} + \frac{25\sqrt{3}}{2}\mathbf{j} + \frac{25}{4}\mathbf{k}$$

This vector is at an angle of 30° to OB so the bearing is 060.

(ii) The particle reaches its maximum height when the $\mathbf{k}$ component is a maximum, so we need to maximise $5\sqrt{5}t - 5t^2$. This has a maximum at $t = \frac{\sqrt{5}}{2}$ which is when the particle is at P.

(iii) It takes $\frac{1}{8}$ seconds for the bullet to reach P. In this time, the particle has travelled a further $\frac{10\sqrt{5}}{8}$ metres horizontally, and just $\frac{5}{64}$ metres vertically.

4 We will use conservation of momentum and Newton's law of restitution.
Let v_a and v_b be the velocities of A and B after the collision.
Relative speed before the collision: $2u - -u = 3u$
Relative speed after the collision: $v_b - v_a$
So by Newton's Law of Restitution, $e = \frac{v_b - v_a}{3u}$.

Conservation of momentum: $2m.2u - mu = 2mv_a + mv_b$ so $3u = 2v_a + v_b$.
Solving simultaneously gives $v_a = u(1-e)$ and $v_b = u(2e+1)$.

Now let q_b be the velocity of B after its collision with the wall.
Thus $q_b = -fv_b = -fu(2e+1)$.
Then there is a second collision with A which has initial velocity $v_a = u(1-e)$.
Let w_a and w_b be the velocities after the second collision. Using momentum and restitution again:

$$w_b - w_a = e(v_a - q_b)$$

$$2mw_a + mw_b = 2mv_a + mq_b$$

Eliminating w_a gives

$$3w_b = 2(e+1)v_a + (1-2e)q_b$$

Substituting in for v_a and q_b gives

$$w_b = \frac{1}{3}(2u(1-e^2) + fu(4e^2 - 1)$$

which can be rearranged to give the required result.

We are required to show that B moves towards the wall for all value of e and f, so we need to show that $\frac{2}{3}(1-e^2) - \frac{1}{3}(1-4e^2)f > 0$.
Consider the case $e = \frac{1}{2}$. Then the expression becomes $\frac{2}{3}\frac{3}{4}$ which is greater than 0.
For $e < \frac{1}{2}$, $1 - e^2 > \frac{3}{4}$ and as $(1-4e^2)$ and f are both less than 1, the expression is greater than 0. When $e > \frac{1}{2}$, $1 - 4e^2$ is negative (and $1 - e^2$ is positive) so the expression is positive.