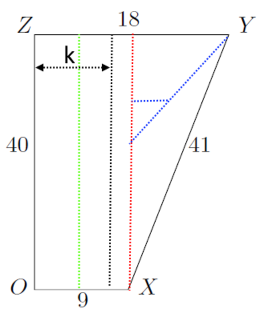
STEP Support Programme

STEP 2 Mechanics Questions: Solutions

These are not fully worked solutions — you need to fill in some gaps. It is a good idea to look at the "Hints" document before this one.

1 **(i)** Divide the lamina into a rectangle and a triangle. Let the Centre of Mass lie on a line a distance k from OZ.



The CoM of the triangle is a horizontal distance 3cm from the red line, so a distance $12-k$ from the CoM of the trapezium. (The CoM of the triangle lies on the intersection of the medians of the triangle.)

The CoM of the rectangle is on the green line which lies half way between O and X and so the CoM of the rectangle is a distance of $k-4.5$ from the CoM of the trapezium.

The rectangle has twice the area so twice the mass - say the rectangle has mass $2M$ and the triangle has mass M. *Alternatively, you could use a mass density ρ instead of the masses of each bit.*

Taking moments about the CoM of the trapezium we have:

$$\begin{aligned} 2\cancel{Mg}(k-4.5) &= \cancel{Mg}(12-k) \\ 2k-9 &= 12-k \\ 3k &= 21 \\ k &= 7 \end{aligned}$$

as required.

Alternatively, we can find the weighted mean of the x coordinates of the centres of mass of the two shapes. This gives us:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} = \frac{2M \times \frac{9}{2} + M \times 12}{3M} = 7$$

(ii) First, work out the areas of each part of the tank:

Front $41d$, Back $40d$, Base $9d$, each side 540.

As the sheet metal has constant density, let the density be ρ, so that the mass of the back is $40d\rho$ etc.

The centre of masses of the faces of the tank are at the following distances from the line OZ:

- LH end: 7 (from part **(i)**)
- RH end: 7
- Front: $\frac{27}{2}$ (i.e. the mean of 18 and 9)
- Back: 0
- Base: $\frac{9}{2}$

We can then find the weighted mean of the x coordinates:

$$
\begin{aligned}
\bar{x_E} &= \frac{2 \times (540\rho) \times 7 + 41d\rho \times \frac{27}{2} + 0 + 9d\rho \times \frac{9}{2}}{1080\rho + 90d\rho} \\
&= \frac{1080 \times 7 + \frac{27}{2} \times (41+3)d}{1080 + 90d} \\
&= \frac{1080 \times 7 + 27 \times 22d}{1080 + 90d} \\
&= \frac{(2^3 \times 3^3 \times 5 \times 7) + (2 \times 3^3 \times 11)d}{(2^3 \times 3^3 \times 5) + (2 \times 3^2 \times 5)d} \\
&= \frac{54(140 + 11d)}{90(12+d)} \\
&= \frac{3(140+11d)}{5(12+d)}
\end{aligned}
$$

Alternatively

Let the centre of mass of the entire tank be a distance q from the back of the tank. Taking moments about the centre of mass:

$$40d\rho q + 9d\rho(q - 4.5) = 2 \times 540\rho(7 - q) + 41d\rho(13.5 - q)$$

For this I have imagined a value for q, such as $q = 6$, which has enabled me to separate clockwise and anti-clockwise motion. If q is different then the signs of some of the terms will change, but everything will still work.

This solves (with a bit of work) to give:

$$q = \frac{3(140+11d)}{5(12+d)} \text{ cm}$$

as required.

When $d = 20$, $q = \frac{27}{4}$, the area of the sheet metal is 2880 and the volume of the tank is 10800. The mass of the metal tank is then 2880ρ and the mass of the water is $10800k\rho$. The centre of mass for the water is a distance of 7 from the line OZ (as in part **(i)**).

Using the weighted mean, the x coordinate of the centre of mass of the filled tank is:

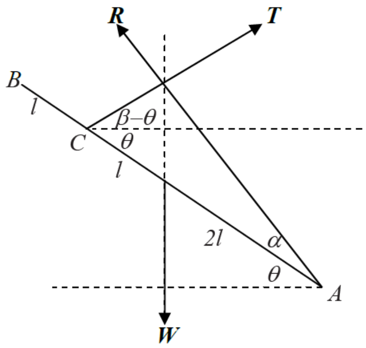
$$
\begin{aligned}
\bar{x_F} &= \frac{2880\rho \times \frac{27}{4} + 10800k\rho \times 7}{2880\rho + 10800k\rho} \\
&= \frac{27 + 105k}{4 + 15k}
\end{aligned}
$$

Alternatively, let p be the distance of the Centre of Mass of the filled tank from the back of the tank. Taking moments:

$$\begin{aligned} 2880\rho\left(p-\frac{27}{4}\right) &= 10800k\rho(7-p) \\ 288p - 72\times 27 &= 1080\times 7k - 1080kp \\ (288+1080k)p &= 72\times 27 + 1080\times 7k \\ (4+15k)p &= 27 + 15\times 7k \\ p &= \frac{27+105k}{4+15k} \end{aligned}$$

At this point, it's a good idea to do a quick sanity check - what happens when the liquid is very much heavier than the tank? (i.e. k is very large.) p is then approximately 7, which is what we would expect as the CoM of the liquid is a distance of 7 from the back of the tank. You can also check what happens when the liquid is weightless, i.e. when $k = 0$.

2 Start, as with virtually all mechanics questions, by drawing a nice clear diagram showing the rod and the forces acting:



Note that the "l's" in the diagram should really be "L's"!

As the rod is held in equilibrium, we know it neither slips nor rotates. We can take moments, and resolve forces horizontally and vertically. Taking moments about A we have:

$$2LW\cos\theta = 3LT\sin\beta$$

so

$$T = \frac{2W\cos\theta}{3\sin\beta}$$

Resolving horizontally:

$$R\cos(\alpha+\theta) = T\cos(\beta-\theta)$$

Substituting in for T:

$$R = \frac{2W\cos\theta\cos(\beta-\theta)}{3\sin\beta\cos(\alpha+\theta)}$$

Resolving vertically:

$$R\sin(\alpha+\theta) + T\sin(\beta-\theta) = W$$

Substituting for R and T:

$$\frac{2W\cos\theta\cos(\beta-\theta)\sin(\alpha+\theta)}{3\sin\beta\cos(\alpha+\theta)} + \frac{2W\cos\theta\sin(\beta-\theta)}{3\sin\beta} = W$$

Cancelling the Ws and multiplying up by the common denominator gives us:

$$\begin{aligned}
2\cos\theta\cos(\beta-\theta)\sin(\alpha+\theta) + 2\cos\theta\sin(\beta-\theta)\cos(\alpha+\theta) &= 3\sin\beta\cos(\alpha+\theta)\\
2\cos\theta\sin\big[(\alpha+\theta)+(\beta-\theta)\big] &= 3\sin\beta\cos(\alpha+\theta)\\
2\cos\theta\sin(\alpha+\beta) &= 3\sin\beta\cos(\alpha+\theta)\\
2\cos\theta\big[\sin\alpha\cos\beta+\sin\beta\cos\alpha\big] &= 3\sin\beta\big[\cos\alpha\cos\theta-\sin\alpha\sin\theta\big]\\
2\cos\theta\sin\alpha\cos\beta + 3\sin\beta\sin\alpha\sin\theta &= \sin\beta\cos\alpha\cos\theta\\
2\frac{\cos\theta\sin\alpha\cos\beta}{\sin\beta\sin\alpha\cos\theta} + 3\frac{\sin\beta\sin\alpha\sin\theta}{\sin\beta\sin\alpha\cos\theta} &= \frac{\sin\beta\cos\alpha\cos\theta}{\sin\beta\sin\alpha\cos\theta}\\
2\frac{\cos\beta}{\sin\beta} + 3\frac{\sin\theta}{\cos\theta} &= \frac{\cos\alpha}{\sin\alpha}\\
2\cos\beta + 3\tan\theta &= \cot\alpha
\end{aligned}$$

Alternatively

Taking moments about A we have:

$$2LW\cos\theta = 3LT\sin\beta \implies T = \frac{2W\cos\theta}{3\sin\beta}$$

Taking moments about C we have:

$$3LR\sin\alpha = LW\cos\theta \implies R = \frac{W\cos\theta}{3\sin\alpha}$$

Resolving parallel to the rod gives:

$$R\cos\alpha = W\sin\theta + T\cos\beta$$

Substituting for R and T in this last equation gives:

$$\begin{aligned}\frac{W\cos\theta}{3\sin\alpha}\times\cos\alpha &= W\sin\theta + \frac{2W\cos\theta}{3\sin\beta}\times\cos\beta\\ \cos\theta\times\frac{\cos\alpha}{\sin\alpha} &= 3\sin\theta + 2\cos\theta\times\frac{\cos\beta}{\sin\beta}\\ \cot\alpha &= 3\tan\theta + 2\cot\beta\end{aligned}$$

Which is a slightly neater method!

Given that $\theta = 30°$ and $\beta = 45°$,

$$\cot\alpha = 3\tan 30 + 2\cot 45 = \sqrt{3} + 2$$

In order to show that $\alpha = 15°$, it is sufficient to show that $\tan 15° = \frac{1}{\sqrt{3}+2}$, which can be done by considering $\tan(60° - 45°)$:

$$\begin{aligned}\tan(60° - 45°) &= \frac{\tan 60° - \tan 45°}{1 + \tan 60°\tan 45°}\\ &= \frac{\sqrt{3}-1}{1+\sqrt{3}}\\ &= \frac{(\sqrt{3}-1)(\sqrt{3}+1)}{(1+\sqrt{3})(\sqrt{3}+1)}\\ &= \frac{2}{4+2\sqrt{3}}\\ &= \frac{1}{2+\sqrt{3}}\end{aligned}$$

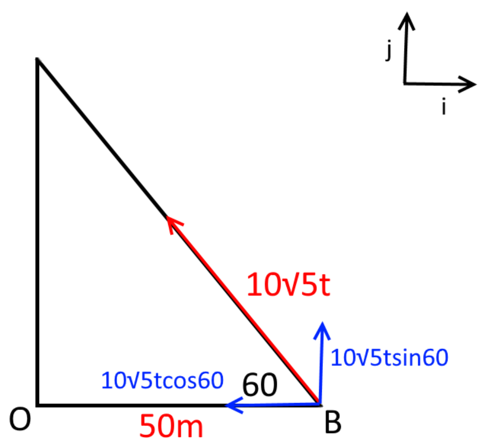
3 Trying to describe motion in 3D can be tricky. Start by drawing a diagram showing the position of O and the initial position of B. One way of thinking about this is to consider a cannon, start by pointing the barrel (horizontally) at an angle of $60°$ to OB, and then move it vertically (point the barrel up) so that it makes an angle of $\arctan \frac{1}{2}$ with the ground.

(i) Start by considering the vertical motion. As $\theta = \arctan \frac{1}{2}$, we have $\tan\theta = \frac{1}{2}$ and by drawing a right-angled triangle we can see that $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$.

Vertical distance travelled at time t:

$$\begin{aligned} s &= 25\sin\theta t - \frac{1}{2}gt^2 \\ &= \frac{25t}{\sqrt{5}} - 5t^2 \\ &= 5\sqrt{5}t - 5t^2 \end{aligned}$$

Horizontally, $s = 25\cos\theta t = 25 \times \frac{2}{\sqrt{5}}\,t = 10\sqrt{5}\,t$, and this is in a direction $60°$ clockwise of the East-West line BO.



We are taking the origin to be at O, so the initial position of the particle is $50\mathbf{i}$. Thus, the position vector relative to O at time t is

$$(50 - 5\sqrt{5}\,t)\,\mathbf{i} + 5\sqrt{15}\,t\,\mathbf{j} + (5\sqrt{5}\,t - 5t^2)\,\mathbf{k}$$

Finding the magnitude squared of this vector gives (taking out a factor of 5 first):

$$\begin{aligned} &25\left((10 - \sqrt{5}\,t)^2 + 15t^2 + (\sqrt{5}\,t - t^2)^2\right) \\ =&25\left(100 - 20\sqrt{5}\,t + 5t^2 + 15t^2 + 5t^2 - 2\sqrt{5}\,t^3 + t^4\right) \\ =&25\left(100 - 20\sqrt{5}\,t + 25t^2 - 2\sqrt{5}\,t^3 + t^4\right) \\ =&25\left(10 - \sqrt{5}\,t + t^2\right)^2 \end{aligned}$$

Hence the distance of the particle from O at time t is $5(t^2 - \sqrt{5}\,t + 10)$ as required.

The distance is shortest when $t^2 - \sqrt{5}\,t + 10$ is at a minimum. We can complete the square to give $t^2 - \sqrt{5}\,t + 10 = (t - \frac{\sqrt{5}}{2})^2 - \frac{5}{4} + 10$ and so $t = \frac{\sqrt{5}}{2}$ at P.

Substituting this value of t into our formula for the position vector gives

$$OP = \frac{75}{2}\,\mathbf{i} + \frac{25\sqrt{3}}{2}\,\mathbf{j} + \frac{25}{4}\,\mathbf{k}$$

By considering the $\mathbf{i}$ and $\mathbf{j}$ components, you can show that this vector is at an angle of $30°$ to OB (measured anticlockwise from OB) so the bearing is $060°$.

(ii) The particle reaches its maximum height when the $\mathbf{k}$ component is a maximum, so we need to maximise $5\sqrt{5}t - 5t^2$. This has a maximum at $t = \frac{\sqrt{5}}{2}$ (this can be found by differentiation or completing the square) which is when the particle is at P.

(iii) We need to find the time it takes the bullet to reach P, and then we can work out how much further the particle has travelled in this time.

The distance of the particle from O is $5(t^2 - \sqrt{5}\,t + 10)5 = 5\left((t - \frac{\sqrt{5}}{2})^2 - \frac{5}{4} + 10\right)$ (from part **(i)**), and at P (when $t = \frac{\sqrt{5}}{2}$) the distance is $5\left(10 - \frac{5}{4}\right) = 5 \times \frac{35}{4}$.

The bullet is travelling at 350ms^{-1}, so it takes $5 \times \frac{35}{4} \div 350 = \frac{1}{8}$ seconds for the bullet to reach P.

At this time (i.e. $t = \frac{\sqrt{5}}{2} + \frac{1}{8} = t_P + \frac{1}{8}$), let the particle be at point P'. The vector $\overrightarrow{PP'}$ is given by:

$$\begin{aligned}
PP' &= \begin{pmatrix} 50 - 5\sqrt{5}\,\left(t_P + \frac{1}{8}\right) \\ 5\sqrt{15}\,\left(t_P + \frac{1}{8}\right) \\ 5\sqrt{5}\,\left(t_P + \frac{1}{8}\right) - 5\left(t_P + \frac{1}{8}\right)^2 \end{pmatrix} - \begin{pmatrix} 50 - 5\sqrt{5}\,t_P \\ 5\sqrt{15}\,t_P \\ 5\sqrt{5}\,t_P - 5t_P^2 \end{pmatrix} \\
&= \begin{pmatrix} -5\sqrt{5} \times \frac{1}{8} \\ 5\sqrt{15} \times \frac{1}{8} \\ 5\sqrt{5} \times \frac{1}{8} - 5\left(\frac{1}{4}t_P + \left(\frac{1}{8}\right)^2\right) \end{pmatrix} \\
&= \begin{pmatrix} -5\sqrt{5} \times \frac{1}{8} \\ 5\sqrt{15} \times \frac{1}{8} \\ 5\sqrt{5} \times \frac{1}{8} - 5\left(\frac{\sqrt{5}}{8} + \left(\frac{1}{8}\right)^2\right) \end{pmatrix} \\
&= \begin{pmatrix} -5\sqrt{5} \times \frac{1}{8} \\ 5\sqrt{15} \times \frac{1}{8} \\ -\frac{5}{64} \end{pmatrix}
\end{aligned}$$

We therefore have

$$\begin{aligned}|PP'|^2 &= \frac{25\times 5}{64} + \frac{25\times 15}{64} + \frac{25}{64^2}\\ &= \frac{125}{64} + \frac{375}{64} + \frac{25}{64^2}\\ &= \frac{500\times 64 + 25}{64^2}\\ &= \frac{32025}{4096} \qquad \text{note that } 64^2 = 2^{12} = 4096\\ &\approx 8\end{aligned}$$

Hence $|PP'| \approx \sqrt{8} \approx 3$.

There is a neater way to do this question by considering the horizontal (**i, j**) and vertical (**k**) movements rather than the three separate directions.

4 Let the velocities of A and B after the collision be v_A and v_B, and we will take the initial direction of A as positive (so the initial velocity of B is $-u$). We then have:

$$\begin{aligned}
\text{Conservation of momentum} \quad &\implies 4mu - mu = 2mv_A + mv_B \\
&\quad\; 3u = 2v_A + v_B \qquad (\mathbf{1}) \\
\text{Newton's experimental law} \quad &\implies v_B - v_A = e(2u + u) \\
&\quad\; v_B - v_A = 3ue \qquad (\mathbf{2})
\end{aligned}$$

Then $(\mathbf{1}) + 2 \times (\mathbf{2}) \implies 3v_B = 3u + 6ue \implies v_B = u(1+2e)$
and $(\mathbf{1}) - (\mathbf{2}) \implies 3v_A = 3u - 3ue \implies v_A = u(1-e)$.

It is very easy to make a sign error with Newton's experimental law / law of restitution! I tend to think of it as "speed of separation $= e\times$ speed of approach" rather than memorising a formula with symbols.

The values of v_A and v_B are both positive, so the positive direction is "towards the wall". Let q_B be the speed of B after the collision with the wall. Newton's experimental law gives us:

$$q_B = -fv_B = -fu(1+2e)$$

Note that the wall is stationary! A sanity check should mean that you realise that the signs of q_B and v_B must be different.

Now we need to consider the second collision between A and B. Let the velocities of A and B after this second collision be w_A and w_B. We have:

$$\begin{aligned}
\text{Conservation of momentum} \quad &\implies 2mv_A + mq_B = 2mw_A + mw_B \\
&\quad\; 2u(1-e) - fu(1+2e) = 2w_A + w_B \qquad (\mathbf{3}) \\
\text{Newton's experimental law} \quad &\implies w_B - w_A = e(v_A - q_B) \\
&\quad\; w_B - w_A = e(u(1-e) + fu(1+2e)) \qquad (\mathbf{4})
\end{aligned}$$

Re-reading the question, we only need to find the velocity of B! So using $(\mathbf{3}) + 2 \times (\mathbf{4})$:

$$\begin{aligned}
3w_B &= \Big(2u(1-e) - fu(1+2e)\Big) + 2\Big(e(u(1-e) + fu(1+2e))\Big) \\
&= 2u(1-e) + 2eu(1-e) - fu(1+2e) + 2fue(1+2e) \\
&= 2u(1-e)[1+e] + fe(1+2e)[2e-1] \\
&= 2u(1-e^2) + fu(4e^2-1)
\end{aligned}$$

and hence we have:

$$w_B = \frac{2}{3}\left(1-e^2\right)u - \frac{1}{3}\left(1-4e^2\right)fu$$

as required.

For B to be moving towards the wall we need $w_B > 0$. Since u and $\frac{1}{3}$ are both positive, we need:

$$\begin{aligned}
2\left(1-e^2\right) - (1-4e^2)f &> 0 \\
\text{i.e.} \quad (4f-2)e^2 + (2-f) &> 0
\end{aligned}$$

This has the form of a quadratic inequality in e. The constant term $2 - f$ will always be positive (as $f \leqslant 1$), whereas the coefficient of e^2 can change sign. There are three cases:

- $f = \frac{1}{2}$ this gives $2 - f$ which is always positive as $f \leqslant 1$, hence $w_B > 0$.
- $\frac{1}{2} < f \leqslant 1$ then $(4f - 2)e^2 > 0$ and $2 - f > 0$ hence $w_B > 0$.
- $0 < f < \frac{1}{2}$ this gives us an inequality of the form $-ke^2 + a > 0$, where $k, a > 0$. Consider a sketch of $y = -ke^2 + a$, there is a maximum point at the origin, at $(0, a)$ and the graph intersects the x axis at $(\pm\sqrt{\frac{a}{k}}, 0)$. Since $0 < e \leqslant 1$, we need $\sqrt{\frac{a}{k}} > 1$ for w_B to be positive for all possible values of e.

This gives us:

$$\begin{aligned}
\sqrt{\frac{a}{k}} &> 1 \\
\sqrt{\frac{2-f}{2-4f}} &> 1 \\
\frac{2-f}{2-4f} &> 1 \\
2 - f &> 2 - 4f \\
3f &> 0
\end{aligned}$$

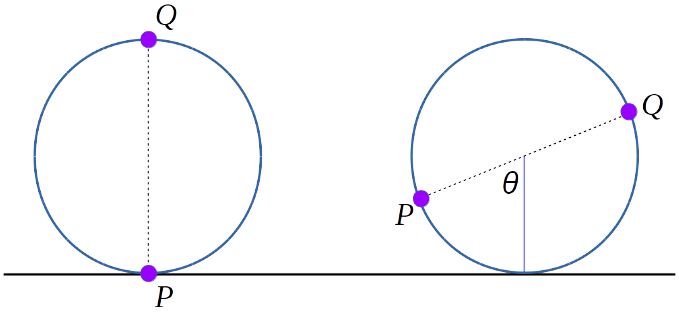
which is true, hence $w_B > 0$ for all possible values of e and f.

Alternatively This is probably a bit neater, but I thought I would leave in the slightly clunky method above.

We need $2\left(1 - e^2\right) - (1 - 4e^2)f > 0$ Consider what happens for different values of e.

- Let $e = \frac{1}{2}$. Then the above becomes $2 \times \frac{1}{4} > 0$.
- Let $0 < e < \frac{1}{2}$. Then we have $1 - e^2 > \frac{3}{4}$ and $1 - 4e^2 < 1$, and $f < 1$. Hence $\left(1 - e^2\right) - (1 - 4e^2)f > 2 \times \frac{3}{4} - 1 \times 1 > 0$.
- Let $\frac{1}{2} < e \leqslant 1$. We have $1 - e^2 \geqslant 0$ and $1 - 4e^2 < 0$ (and $f > 0$) hence $\left(1 - e^2\right) - (1 - 4e^2)f > 0$.

5 Here we are considering a combination of the rotation about the centre and movement of the hoop horizontally. First, draw a diagram:



The position vector of point P is:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a\sin\theta + a\theta \\ a(1-\cos\theta) \end{pmatrix} = \begin{pmatrix} a\left(\theta - \sin\theta\right) \\ a(1-\cos\theta) \end{pmatrix}$$

and the velocity vector is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a\left(\dot{\theta} - \cos\theta \times \dot{\theta}\right) \\ a\sin\theta \times \dot{\theta} \end{pmatrix} = \begin{pmatrix} a\dot{\theta}\left(1-\cos\theta\right) \\ a\dot{\theta}\sin\theta \end{pmatrix}$$

The speed is given by

$$\begin{aligned} \sqrt{\dot{x}^2 + \dot{y}^2} &= \sqrt{\left(a\dot{\theta}\right)^2 \left((1-\cos\theta)^2 + \sin^2\theta\right)} \\ &= \sqrt{\left(a\dot{\theta}\right)^2 \left(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right)} \\ &= a|\dot{\theta}|\sqrt{2(1-\cos\theta)} \\ &= \sqrt{2}a|\dot{\theta}|\left(1-\cos\theta\right)^{1/2} \end{aligned}$$

When P is at an angle θ with the downwards vertical, Q is at an angle of $\theta + \pi$. Hence the speed of Q is $\sqrt{2}a|\dot{\theta}|\left(1-\cos\left(\theta+\pi\right)\right)^{1/2} = \sqrt{2}a|\dot{\theta}|\left(1+\cos\theta\right)^{1/2}$ (using $\cos(\theta+\pi) = -\cos\theta$). The total kinetic energy is:

$$\begin{aligned} &\tfrac{1}{2}m \times 2a^2|\dot{\theta}|^2\left(1-\cos\theta\right) + \tfrac{1}{2}m \times 2a^2|\dot{\theta}|^2\left(1+\cos\theta\right) \\ =&ma^2\dot{\theta}^2\left(1-\cos\theta\right) + ma^2\dot{\theta}^2\left(1+\cos\theta\right) \\ =&2ma^2\dot{\theta}^2 \end{aligned}$$

Since the only external forces are gravity and the vertical reaction force, energy is conserved, i.e. the sum of the kinetic energy and potential energy of the two particles is constant. The potential energy is given by:

$$mg(a - a\cos\theta) + mg(a + a\cos\theta) = 2mga$$

Hence $2mga + 2ma^2\dot{\theta}^2$ is a constant, and since m, a and g are constant we must have $\dot{\theta}$ constant and so the hoop rolls with constant speed.

6 The key thing here is a system of labelling the velocities which is easy to follow. I will take the initial velocities as u_1, u_2 and u_3, the velocities after the first collision as v_1, v_2 and v_3 and the velocities after the second collision as w_1, w_2 and w_3. A lot of these will be zero, and we have $u_1 = v$ and $w_3 = v$.

For the first collision we have:

- $m_1 v = m_2 v_2$
- $v_2 = ev$

From these two equations we have $m_1 v = m_2 \times ev \implies m_2 e = m_1$.

For the second collision we have:

- $m_2 v_2 = m_2 w_2 + m_3 v$
- $v - w_2 = e' v_2$

We want to write e' in terms of the masses. We have:

$$\begin{aligned}
e' v_2 &= v - w_2 \\
e' v_2 &= v - \frac{m_2 v_2 - m_3 v}{m_2} \\
e' \times ev &= v - \frac{m_2 \times ev - m_3 v}{m_2} \\
e' e &= 1 - \frac{m_2 e - m_3}{m_2} \\
e' e &= \frac{m_2 - m_2 e + m_3}{m_2} \\
e' &= \frac{m_2 - m_2 e + m_3}{m_2 e} \\
e' &= \frac{m_2 - m_1 + m_3}{m_1}
\end{aligned}$$

The coefficient of restitution lies between 0 and 1. We therefore need:

$$\begin{aligned}
0 \leqslant &\frac{m_2 + m_3 - m_1}{m_1} \leqslant 1 \\
0 \leqslant &\, m_2 + m_3 - m_1 \leqslant m_1 \\
m_1 \leqslant &\, m_2 + m_3 \leqslant 2m_1
\end{aligned}$$

The final (kinetic) energy of the system is

$$\begin{aligned}
&\tfrac{1}{2} m_2 w_2^2 + \tfrac{1}{2} m_3 v^2 \\
=&\tfrac{1}{2} \frac{(m_2 v_2 - m_3 v)^2}{m_2} + \tfrac{1}{2} m_3 v^2 \\
=&\tfrac{1}{2} \frac{(m_1 v - m_3 v)^2}{m_2} + \tfrac{1}{2} m_3 v^2 \\
=&\tfrac{1}{2} v^2 \left(\frac{(m_1 - m_3)^2}{m_2} + m_3 \right)
\end{aligned}$$

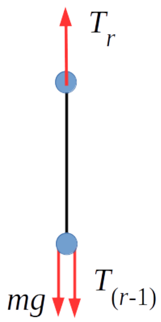
The condition $m_1 \leqslant m_2 + m_3 \leqslant 2m_1$ can be written as $m_1 - m_3 \leqslant m_2 \leqslant 2m_1 - m_3$. The minimum final energy will be when m_2 is at it's maximum, so is:

$$\begin{aligned}\tfrac{1}{2}v^2\left(\frac{(m_1-m_3)^2}{2m_1-m_3}+m_3\right) &= \tfrac{1}{2}v^2\left(\frac{(m_1-m_3)^2+m_3(2m_1-m_3)}{2m_1-m_3}\right)\\ &= \tfrac{1}{2}v^2\frac{m_1^2}{2m_1-m_3}\end{aligned}$$

It appears as if the minimum value of m_2 should be $m_1 - m_3$, but from the first collision we have $e = \frac{m_1}{m_2}$, hence $m_2 \geqslant m_1$ and the minimum value of m_2 is m_1. The maximum energy is hence:

$$\tfrac{1}{2}v^2\left(\frac{(m_1-m_3)^2}{m_1}+m_3\right) = \tfrac{1}{2}v^2\left(\frac{m_1^2+m_3^2-m_1m_3}{m_1}\right)$$

7 We have n light strings. Considering the "r^{th}" string we have the following situation:



We therefore have $T_r = mg + T_{r-1}$. (A quick sanity check shows that this makes sense, you would expect the string at the top to have the largest tension, and string 1 — the bottom string — will have tension mg as it is just holding up the bottom particle).

Using this recursive formula, with $T_1 = mg$, gives us $T_r = rmg$.

Hooke's law gives us $T_r = \dfrac{\lambda x_r}{l}$. This gives $\dfrac{\lambda x_r}{l} = rmg \implies x_r = \dfrac{rmgl}{\lambda}$. The total length of the long string is therefore:

$$\begin{aligned}\sum_{r=1}^{n}(l + x_r) &= \sum_{r=1}^{n}\left(l + \frac{rmgl}{\lambda}\right)\\ &= nl + \frac{mgl}{\lambda}\sum_{r=1}^{n} r\\ &= nl + \frac{mgl \times n(n+1)}{2\lambda}\end{aligned}$$

The elastic energy stored is given by:

$$\begin{aligned}\sum_{r=1}^{n}\frac{\lambda x_r^2}{2l} &= \sum_{r=1}^{n}\frac{\lambda}{2l} \times \left(\frac{rmgl}{\lambda}\right)^2\\ &= \frac{m^2g^2l}{2\lambda}\sum_{r=1}^{n} r^2\\ &= \frac{m^2g^2l}{2\lambda} \times \tfrac{1}{6}n(n+1)(2n+1)\end{aligned}$$

For the uniform heavy rope, we can think of this as being a very large number of very short strings with equal weight particles attached. We let $M = nm$ and $L_0 = nl$, and then consider the limit of the above answers as $n \to \infty$.

We have:

$$\begin{aligned}
L &= \lim_{n\to\infty}\left(nl + \frac{mgl\times n(n+1)}{2\lambda}\right)\\
&= \lim_{n\to\infty}\left(L_0 + \frac{M}{n}\times\frac{g}{\lambda}\times\frac{L_0}{n}\times\frac{n(n+1)}{2}\right)\\
&= \lim_{n\to\infty}\left(L_0 + \frac{MgL_0}{\lambda}\times\tfrac{1}{2}\left(1+\frac{1}{n}\right)\right)\\
&= L_0 + \frac{MgL_0}{2\lambda}\\
L &= L_0\left(1+\frac{Mg}{2\lambda}\right) \qquad (*)
\end{aligned}$$

The elastic energy is given by:

$$\begin{aligned}
\text{E.P.E.} &= \lim_{n\to\infty}\left(\frac{m^2g^2l}{2\lambda}\times\tfrac{1}{6}n(n+1)(2n+1)\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{12\lambda}\times\left(\frac{M}{n}\right)^2\times g^2\times\frac{L_0}{n}\times n(n+1)(2n+1)\right)\\
&= \frac{M^2g^2L_0}{12\lambda}\lim_{n\to\infty}\frac{2n^3+3n^2+n}{n^3}\\
&= \frac{M^2g^2L_0}{12\lambda}\lim_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\\
&= \frac{M^2g^2L_0}{6\lambda}
\end{aligned}$$

The question asks us to find this in terms of L, L_0 and λ, so we need to eliminate M. Using $(*)$ we have $M = \dfrac{2\lambda(L-L_0)}{gL_0}$, and so we have:

$$\begin{aligned}
\text{E.P.E} &= \frac{M^2g^2L_0}{6\lambda}\\
&= \frac{4\lambda^2(L-L_0)^2}{g^2L_0^2}\times\frac{g^2L_0}{6\lambda}\\
&= \frac{2\lambda(L-L_0)^2}{3L_0}
\end{aligned}$$