STEP Support Programme

STEP 3 Mechanics: Solutions

1 **(i)** Let the velocity of the first mass after the collision be $\mathbf{v_1}$ and the velocity of the second mass be $\mathbf{v_2} = v_2\mathbf{n}$.
Conservation of momentum gives us:

$$m\mathbf{u} = m\mathbf{v_1} + m\mu v_2\mathbf{n} \implies \mathbf{u} = \mathbf{v_1} + \mu v_2\mathbf{n}$$

Newton's experimental law applies to the speeds in the direction of the "line of centres" of the two discs when they meet. Since the second disc starts at rest and moves in the direction given by $\mathbf{n}$ after the collision then the direction in which Newton's experimental law applies is the direction represented by $\mathbf{n}$. We need to find the component of speed in the same direction as $\mathbf{n}$ for the first particle before and after the collision. These are given by $\mathbf{u}\cdot\mathbf{n}$ and $\mathbf{v_1}\cdot\mathbf{n}$ respectively (see the hints for an explanation). The collision is elastic (e=1) so we have:

$$\mathbf{u}\cdot\mathbf{n} = v_2 - \mathbf{v_1}\cdot\mathbf{n}$$

Substituting $\mathbf{v_1} = \mathbf{u} - \mu v_2\mathbf{n}$ (from conservation of momentum) we have:

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{n} &= v_2 - \mathbf{v_1}\cdot\mathbf{n}\\
\mathbf{u}\cdot\mathbf{n} &= v_2 - (\mathbf{u} - \mu v_2\mathbf{n})\cdot\mathbf{n}\\
\mathbf{u}\cdot\mathbf{n} &= v_2 - \mathbf{u}\cdot\mathbf{n} + \mu v_2\\
2\mathbf{u}\cdot\mathbf{n} &= (1+\mu)v_2\\
\implies v_2 &= \frac{2\mathbf{u}\cdot\mathbf{n}}{1+\mu}
\end{aligned}$$

(ii) If the two discs have equal kinetic energy then we have:

$$\tfrac{1}{2}m|\mathbf{v_1}|^2 = \tfrac{1}{2}\mu m v_2^2 \implies |\mathbf{v_1}|^2 = \mu v_2^2$$

Using $\mathbf{v_1} = \mathbf{u} - \mu v_2\mathbf{n}$ from part **(i)** we have:

$$\begin{aligned}
(\mathbf{u} - \mu v_2\mathbf{n})\cdot(\mathbf{u} - \mu v_2\mathbf{n}) &= \mu v_2^2\\
\mathbf{u}\cdot\mathbf{u} - 2\mu v_2\mathbf{u}\cdot\mathbf{n} + \mu^2 v_2^2\mathbf{n}\cdot\mathbf{n} &= \mu v_2^2\\
|\mathbf{u}|^2 - 2\mu v_2\mathbf{u}\cdot\mathbf{n} &= \mu(1-\mu)v_2^2\\
|\mathbf{u}|^2 - 2\mu \times \frac{2\mathbf{u}\cdot\mathbf{n}}{1+\mu} \times \mathbf{u}\cdot\mathbf{n} &= \mu(1-\mu)\frac{4(\mathbf{u}\cdot\mathbf{n})^2}{(1+\mu)^2}\\
(1+\mu)^2|\mathbf{u}|^2 - 4\mu(1+\mu)(\mathbf{u}\cdot\mathbf{n})^2 &= 4\mu(1-\mu)(\mathbf{u}\cdot\mathbf{n})^2\\
(1+\mu)^2|\mathbf{u}|^2 &= 4\mu\Big[(1-\mu)+(1+\mu)\Big](\mathbf{u}\cdot\mathbf{n})^2\\
(1+\mu)^2|\mathbf{u}|^2 &= 8\mu(\mathbf{u}\cdot\mathbf{n})^2
\end{aligned}$$

We have $\mathbf{u}\cdot\mathbf{n} = |\mathbf{u}||\mathbf{n}|\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{n}$, and so:

$$\begin{aligned}
(1+\mu)^2|\mathbf{u}|^2 &= 8\mu|\mathbf{u}|^2\cos^2\theta\\
\implies \cos^2\theta &= \frac{(1+\mu)^2}{8\mu}
\end{aligned}$$

We know that $0 \leqslant \cos^2\theta \leqslant 1$ and so we have:

$$0 \leqslant \frac{(1+\mu)^2}{8\mu} \leqslant 1$$

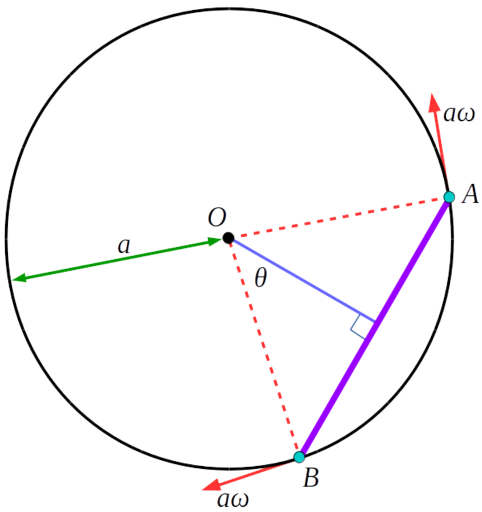
Since $\mu > 0$ (otherwise the second disc has zero or negative mass) and $(1+\mu)^2 > 0$ we have:

$$\begin{aligned}
\frac{(1+\mu)^2}{8\mu} &\leqslant 1 \\
(1+\mu)^2 &\leqslant 8\mu \\
1 + 2\mu + \mu^2 &\leqslant 8\mu \\
\mu^2 - 6\mu + 1 &\leqslant 0 \\
(\mu-3)^2 - 9 + 1 &\leqslant 0 \\
(\mu-3)^2 &\leqslant 8
\end{aligned}$$

Sketching a quick graph of $(\mu-3)^2$ or $(\mu-3)^2 - 8$ leads to:

$$\begin{aligned}
(\mu-3)^2 &\leqslant 8 \\
\implies -\sqrt{8} \leqslant \mu - 3 &\leqslant \sqrt{8} \\
\implies 3 - \sqrt{8} \leqslant \mu &\leqslant 3 + \sqrt{8}
\end{aligned}$$

2 Start by drawing a diagram.



The hoop is smooth, so energy is conserved, i.e. the kinetic energy of the two particles added to the energy stored in the spring is constant. This gives us:

$$2 \times \tfrac{1}{2}m\left(a\dot{\theta}\right)^2 + mk^2a^2\left(\theta - \alpha\right)^2 = c \quad \Longrightarrow$$
$$\dot{\theta}^2 + k^2\left(\theta - \alpha\right)^2 = c'$$

If we differentiate this with respect to time we have:

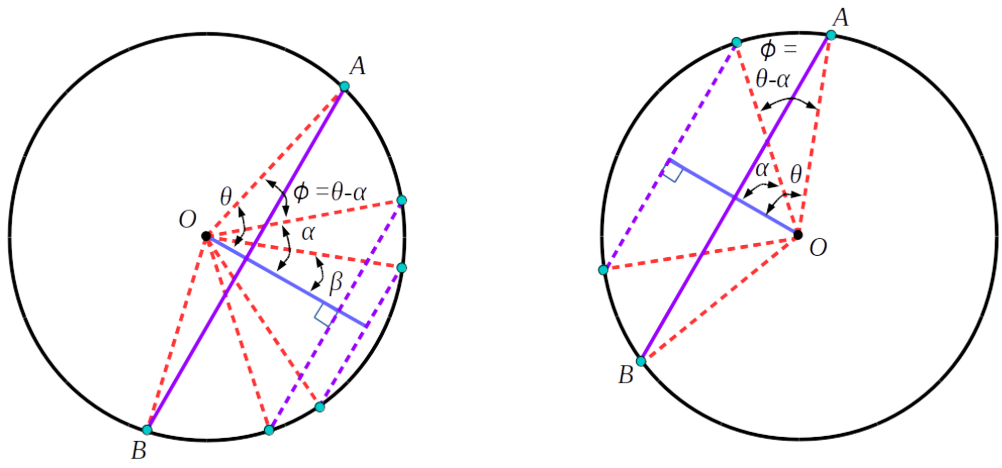
$$2\dot{\theta}\ddot{\theta} + 2k^2\dot{\theta}\left(\theta - \alpha\right) = 0$$

and, assuming that $\dot{\theta}$ is not identically zero:

$$\ddot{\theta} + k^2\left(\theta - \alpha\right) = 0 \quad \Longrightarrow$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\theta - \alpha\right) + k^2\left(\theta - \alpha\right) = 0$$

If we let $\theta - \alpha = \phi$, then we get the equation $\ddot{\phi} = -k^2\phi$, i.e. Simple Harmonic Motion.

The different cases that occur depend on whether the spring passes through to the other "side" of the circle or not. This depends on whether the original compressed spring has enough energy to go "over the top" (i.e. to reach $\theta = \frac{\pi}{2}$). The diagram on the next page shows the "compressed spring" (when $\theta = \beta$), the "natural length" (when $\theta = \alpha$) and what happens as the beads go "over the top".



Case 1 Assume that $\theta < \frac{\pi}{2}$ throughout the motion, which means that the spring stays on the same "side" of the circle. Considering:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\theta - \alpha) + k^2(\theta - \alpha) = 0$$

we have Simple Harmonic Motion with period $\dfrac{2\pi}{k}$. Since the spring starts in compression we have $\beta < \alpha$ and $\theta - \alpha = (\beta - \alpha)\cos kt$. Hence the maximum value of $\theta - \alpha$ is $\alpha - \beta$, so the maximum value of θ is $\alpha + \alpha - \beta$.

Then:

$$\begin{aligned}\theta < \frac{\pi}{2} &\implies 2\alpha - \beta < \frac{\pi}{2}\\ &\implies \beta > 2\alpha - \frac{\pi}{2}\end{aligned}$$

This makes sense when you think about the system. The *smaller* β is then the more "squished" the spring is, so it will have a greater initial potential energy, and so the beads are more likely to go "over the top".

Case 2 When $\beta = 2\alpha - \frac{\pi}{2}$, then we can have $\theta = \frac{\pi}{2}$, which will happen when $\cos kt = -1$. At this point we have $\dot{\theta} = k(\beta - \alpha)\sin kt$ and as $\sin kt = 0$ when $\cos kt = -1$ we have $\dot{\theta} = 0$. At this point the beads are stationary on either side of a diameter of the circle, so the spring is pulling them both "inwards", so they stay where they are and there are no oscillations.

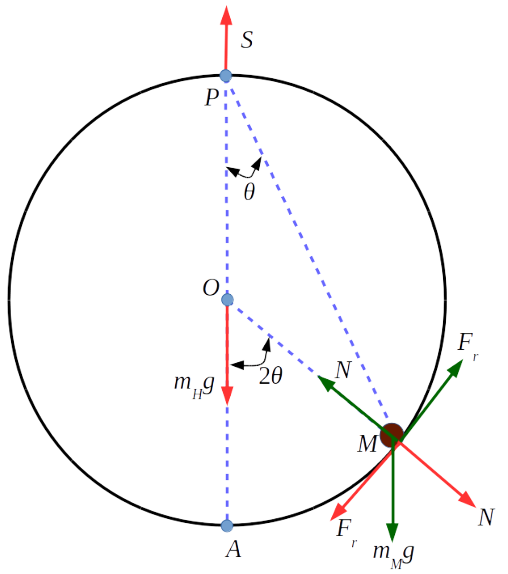
Case 3 If $\beta < 2\alpha - \frac{\pi}{2}$, then at some time ($t_1$ say) we have $\theta = \frac{\pi}{2}$, when we have:

$$\begin{aligned}\frac{\pi}{2} - \alpha = (\beta - \alpha)\cos kt_1 \quad &\implies\\ \cos kt_1 = \frac{\frac{\pi}{2} - \alpha}{\beta - \alpha} \quad &\implies\\ t_1 = \frac{1}{k}\arccos\frac{\frac{\pi}{2} - \alpha}{\beta - \alpha}&\end{aligned}$$

Note that as $\beta < 2\alpha - \frac{\pi}{2}$ we have $\cos kt_1 = \dfrac{\frac{\pi}{2} - \alpha}{\beta - \alpha} > \dfrac{\frac{\pi}{2} - \alpha}{2\alpha - \frac{\pi}{2} - \alpha} = -1$ and so $\dot{\theta} = k\sin(kt_1) \neq 0$ and the beads pass "over the top".

Once we pass $\theta = \frac{\pi}{2}$, the beads pass "over the top" and the motion is repeated on the other side (so the spring will compress back down again, then extend and pass back to the original side). Hence the time period is $4 \times t_1 = \dfrac{4}{k} \arccos \dfrac{\frac{\pi}{2} - \alpha}{\beta - \alpha}$.

3 The diagram below shows the forces acting on the hoop (in red) and the forces acting on the mouse (in green). Let $\angle AOM = 2\theta$, which means that $\angle APM = \theta$. The suspension force from the axis on hoop is S, and N is the normal reaction force and F_r is the frictional force between the hoop and the mouse. Let m_H be the mass of the hoop and m_M be the mass of the mouse. The hoop is "free to rotate" so there is no friction between the hoop and the axis through P.



If the system is in equilibrium then there is no "turning force" acting on the hoop, i.e. the moment about P is zero. The "lines of action" of S and the weight pass through P, so have zero moment about P. The net moment about P is:

$$|PM| \times F_r \cos\theta - |PM| \times N \sin\theta = 0 \qquad \Longrightarrow$$

$$F_r = N \tan\theta \tag{1}$$

When the mouse is running at a constant speed u then its tangential acceleration is zero and the radial acceleration is given by $\frac{u^2}{a}$. The equations of motion for the mouse are therefore:

$$F_r - m_M g \sin 2\theta = 0 \tag{2}$$

$$N - m_M g \cos 2\theta = \frac{m_M u^2}{a} \tag{3}$$

Using (1) and (2) gives:

$$\begin{aligned} N &= \frac{1}{\tan\theta} \times m_M g \sin 2\theta \\ &= \frac{\cos\theta}{\sin\theta} \times 2m_M g \sin\theta \cos\theta \\ &= 2m_M g \cos^2\theta \end{aligned}$$

Then substituting this into (3) gives:

$$\begin{aligned} \frac{m_M u^2}{a} &= 2m_M g \cos^2\theta - -m_M g \cos 2\theta \\ \frac{u^2}{a} &= 2g\cos^2\theta - g\cos 2\theta \\ u^2 &= ag\left(2\cos^2\theta - (2\cos^2\theta - 1)\right) \\ u^2 &= ag \end{aligned}$$

Hence if the mouse chooses a constant speed of $u = \sqrt{ag}$ it can run whilst the hoop stays with diameter POA vertical.

If $u^2 = ag$, then we have $F_r = m_M g \sin 2\theta$ and:

$$\begin{aligned} N &= m_M g \cos 2\theta + \frac{m_M u^2}{a} \\ &= m_M g \cos 2\theta + \frac{m_M ag}{a} \\ &= m_M g \left(\cos 2\theta + 1\right) \\ &= 2m_M g \cos^2 \theta \end{aligned}$$

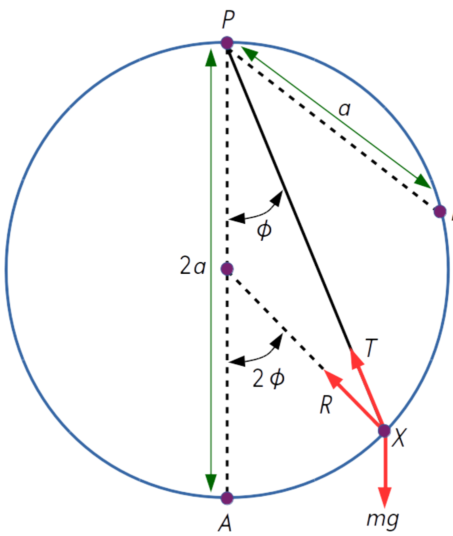
We can write F_r as $F_r = 2m_M g \cos^2 \theta \tan \theta$.

Limiting friction occurs when $F_r = \mu N$, i.e.:

$$\begin{aligned} 2m_M g \cos^2 \theta \tan \theta &= \mu \times 2m_M g \cos^2 \theta \\ \tan \theta &= \mu \\ \theta &= \arctan \mu \end{aligned}$$

If angle θ goes beyond this, i.e. when $\angle AOM = 2\theta$ exceeds $2 \arctan \mu$ then the mouse will start to slip and the hoop will start to rotate in the opposite direction to the mouse (in the diagram at the start of this question the hoop will swing to the left).

4 The diagram below shows the initial position of the ring (A), the point at which the string becomes slack (B) and a general position of the ring. The forces acting on the ring are shown in red.



A very important point to note is that the angle at the centre is 2ϕ, so for the circular equations of motion we need to replace θ with 2ϕ (and $\dot{\theta} = (\dot{2\phi})$ etc.).

Let the Potential Energy of the ring at its starting position (i.e. point A) be 0. As the points P, B and the centre of the hoop form an equilateral triangle we have $\angle APB = \frac{\pi}{3}$. This means that the vertical height of B above A is $a + \frac{1}{2}a = \frac{3}{2}a$.

We are told that the ring starts from rest and that it comes to rest just as the string becomes slack. Using conservation of energy at the points A and B we have:

$$\frac{\lambda a^2}{2a} = \tfrac{3}{2}a \times mg \qquad \Longrightarrow$$
$$\lambda = 3mg$$

Hence the modulus of elasticity is $\lambda = 3mg$.

When the string makes an angle ϕ to the downwards vertical, there is potential energy, elastic potential energy and kinetic energy to consider. At this point, the length of the string is given by:

$$\begin{aligned} L^2 &= a^2 + a^2 - 2a^2 \cos(\pi - 2\phi) \\ &= 2a^2 + 2a^2 \cos(2\phi) \\ &= 2a^2 \Big(1 + \cos(2\phi)\Big) \\ &= 2a^2 \times 2\cos^2\phi \qquad \Longrightarrow \\ L &= 2a\cos\phi \end{aligned}$$

Therefore the extension in the string at this point is $x = 2a\cos\phi - a$.

Using conservation of energy at the point when the string makes angle ϕ with the downwards vertical and at the point when the string becomes slack gives:

$$\begin{aligned}
\tfrac{3}{2}mga &= mg\left(a - a\cos 2\phi\right) + \tfrac{1}{2}m\left(a(2\dot{\phi})\right)^2 + \frac{\lambda\left(2a\cos\phi - a\right)^2}{2a} \\
\tfrac{3}{2}mga &= mga\left(1 - \cos 2\phi\right) + 2ma^2\dot{\phi}^2 + \frac{3mg \times a^2\left(2\cos\phi - 1\right)^2}{2a} \\
3g &= 2g\left(1 - \cos 2\phi\right) + 4a\dot{\phi}^2 + 3g\left(2\cos\phi - 1\right)^2 \\
3g &= 2g\left(2 - 2\cos^2\phi\right) + 4a\dot{\phi}^2 + 3g\left(4\cos^2\phi - 4\cos\phi + 1\right) \\
3g &= 8g\cos^2\phi - 12g\cos\phi + 7g + 4a\dot{\phi}^2 \\
0 &= 8g\cos^2\phi - 12g\cos\phi + 4g + 4a\dot{\phi}^2 \qquad \Longrightarrow \\
4a\dot{\phi}^2 &= -8g\cos^2\phi + 12g\cos\phi - 4g \qquad (*)
\end{aligned}$$

The tension in the string at the general point is $T = \dfrac{\lambda x}{l} = \dfrac{3mg\left(2a\cos\phi - a\right)}{a} = 3mg\left(2\cos\phi - 1\right)$.

Using $F = ma$ radially for the ring gives:

$$ma(2\dot{\phi})^2 = R - mg\cos 2\phi + 3mg\left(2\cos\phi - 1\right)\cos\phi$$

Rearranging gives:

$$R = 4ma\dot{\phi}^2 + mg\cos 2\phi - 3mg\cos\phi\left(2\cos\phi - 1\right)$$

Using $(*)$ to eliminate $\dot{\phi}^2$ results in:

$$\begin{aligned}
R &= m\Big(12g\cos\phi - 8g\cos^2\phi - 4g\Big) + mg\cos 2\phi - 3mg\cos\phi\left(2\cos\phi - 1\right) \\
R &= mg\Big[12\cos\phi - 8\cos^2\phi - 4 + 2\cos^2\phi - 1 - 3\left(2\cos^2\phi - \cos\phi\right)\Big] \\
R &= mg\Big[-12\cos^2\phi + 15\cos\phi - 5\Big]
\end{aligned}$$

This looks like the given result, but is the negative of such. It might be that this R is always negative, so to find the magnitude we would need $|R| = -R$.

Completing the square on our expression for R gives:

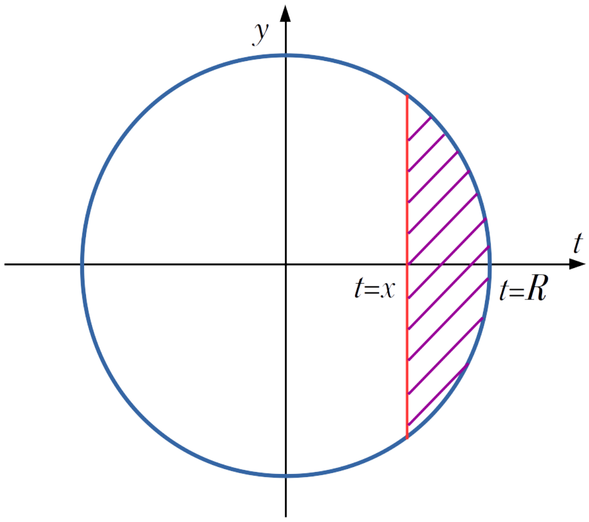
$$\begin{aligned}
R &= mg\Big[-12\cos^2\phi + 15\cos\phi - 5\Big] \\
&= mg\Big[-12\left(\cos^2\phi - \tfrac{15}{12}\cos\phi\right) - 5\Big] \\
&= mg\Big[-12\Big(\left(\cos\phi - \tfrac{5}{8}\right)^2 - \left(\tfrac{5}{8}\right)^2\Big) - 5\Big] \\
&= mg\Big[-12\Big(\left(\cos\phi - \tfrac{5}{8}\right)^2\Big) + 12 \times \left(\tfrac{5}{8}\right)^2 - 5\Big] \\
&= mg\Big[-12\Big(\left(\cos\phi - \tfrac{5}{8}\right)^2\Big) - \tfrac{5}{16}\Big]
\end{aligned}$$

This is always negative, so R can never be zero. The magnitude of the reaction force is therefore

$$R = mg\Big[12\cos^2\phi - 15\cos\phi + 5\Big]$$

and the magnitude is always greater than or equal to $\frac{5mg}{16}$.

5 You can find the volume of the sphere below the surface of the liquid by considering a volume of revolution. Since x will be involved in one of the limits, it is best to use $t^2 + y^2 = R^2$ or similar. The picture below shows the section we are revolving to find the required volume.



The required volume is given by:

$$\pi \int_x^R R^2 - t^2 \, \mathrm{d}t = \pi \Big[R^2 t - \tfrac{1}{3}t^3 \Big]_x^R$$
$$= \pi \Big[R^3 - \tfrac{1}{3}R^3 \Big] - \pi \Big[R^2 x - \tfrac{1}{3}x^3 \Big]$$
$$= \frac{\pi}{3} \Big[2R^3 - 3R^2 x + x^3 \Big]$$

The weight of the sphere (which has density ρ_s) is:

$$W = \frac{4}{3}\pi R^3 \rho_s g$$

and the upward force is given by:

$$U = \frac{\pi}{3}\Big[2R^3 - 3R^2 x + x^3 \Big] \rho g$$

Using "$F = ma$", and considering the fact that as the sphere moves downwards into the liquid, x decreases and U will increase (and act to increase x again) we have:

$$\frac{4}{3}\pi R^3 \rho_s \ddot{x} = \frac{\pi}{3}\Big[2R^3 - 3R^2 x + x^3 \Big] \rho g - \frac{4}{3}\pi R^3 \rho_s g$$
$$\implies \frac{4}{3}\pi R^3 \rho_s \left(\ddot{x} + g \right) = \frac{\pi}{3}\Big[2R^3 - 3R^2 x + x^3 \Big] \rho g$$
$$\implies 4R^3 \rho_s \left(\ddot{x} + g \right) = \Big[2R^3 - 3R^2 x + x^3 \Big] \rho g \qquad (\dagger)$$

At equilibrium we have $\ddot{x} = 0$ (and we are told this happens when $x = \frac{1}{2}R$, and so:

$$\begin{aligned} 4R^3\rho_s g &= \left[2R^3 - 3R^2 \times \tfrac{1}{2}R + \left(\tfrac{1}{2}R\right)^3\right]\rho g \\ 4\rho_s &= \left[2 - \tfrac{3}{2} + \tfrac{1}{8}\right]\rho \\ 4\rho_s &= \frac{5}{8}\rho \\ \rho_s &= \frac{5}{32}\rho \end{aligned}$$

When the oscillations are small then x remains close to $\frac{1}{2}R$. Let $x = u + \frac{1}{2}R$, where $|u| << R$[1].

Substituting $x = u + \frac{1}{2}R$ and $\rho_s = \frac{5}{32}\rho$ into (†) gives:

$$\begin{aligned} 4R^3 \times \frac{5}{32}\rho\left(\ddot{u} + g\right) &= \left[2R^3 - 3R^2\left(\tfrac{1}{2}R + u\right) + \left(\tfrac{1}{2}R + u\right)^3\right]\rho g \\ R^3 \times \frac{5}{8}\rho\left(\ddot{u} + g\right) &= \left[2R^3 - \tfrac{3}{2}R^3 - 3R^2u + \tfrac{1}{8}R^3 + \tfrac{3}{4}R^2u + \tfrac{3}{2}Ru^2 + u^3\right]\rho g \\ 5R^3\rho(\ddot{u} + g) &= \left[16R^3 - 12R^3 - 24R^2u + R^3 + 6R^2u + 12Ru^2 + u^3\right]\rho g \\ 5R^3\rho\ddot{u} + \cancel{5R^3\rho g} &= \cancel{5R^3\rho g} - 18R^2u\rho g + 12Ru^2\rho g + u^3\rho g \\ 5\ddot{u} &= -18g\frac{u}{R} + 12g\left(\frac{u}{R}\right)^2 + g\left(\frac{u}{R}\right)^3 \end{aligned}$$

Then, since $\left|\frac{u}{R}\right| << 1$ we can ignore the squared and higher terms so:

$$\ddot{u} = -\frac{18g}{5R}u$$

which is the standard equation for simple harmonic motion. The oscillations of the motion have period:

$$2\pi\sqrt{\frac{5R}{18g}} = \pi\sqrt{\frac{20R}{18g}} = \frac{\pi}{3}\sqrt{\frac{10R}{g}}$$

(Any of these answers, or any other equivalent ones, would have been fine!)

[1] This means that "u is very much smaller than R". It can perhaps more usefully be written as $\left|\frac{u}{R}\right| << 1$.