STEP Support Programme

STEP 3 Mechanics Topic Notes

Further Collisions

In STEP 3 you might be asked questions about *oblique* impacts (which usually means you need to consider momentum and Newton's Experiment Law in two perpendicular directions).

Impact between a smooth sphere and a fixed surface (wall)

- The impulse (change in momentum) on the sphere acts perpendicularly to the wall
- Newton's Experimental Law acts on the component of velocity of the sphere which is perpendicular to the wall
- The component of the sphere's velocity parallel to the wall is unchanged

Impact between two smooth spheres

- The impulse acts along the line joining the centres of the spheres
- The components of the velocities perpendicular to the line joining the centres is unchanged (acting tangentially)
- Conservation of momentum and Newton's Experimental Law apply to the velocities parallel to the line joining the centres

Centre of mass

The effect of gravity on an object can be thought of as a single force acting at the object's centre of mass.

If you have a *uniform* rigid body then the centre of mass will lie on any lines of symmetry (for a lamina) or planes of symmetry (for a 3-D shape). "Uniform" in this case means that the mass is evenly spread out throughout the rigid body.

If you have a system of particles, or a rigid body made up of several rigid bodies whose centres of mass are known then the coordinates of the centre of mass of the whole system are the weighted mean of the coordinates of the separate parts.

For example the x coordinate will be given by:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

Or if you describe the position of the individual masses with vectors, $\mathbf{r}_i$:

$$\bar{\mathbf{r}} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$$

Using integration to find centres of mass

Uniform lamina

Consider a uniform lamina bounded by the x-axis, $x = a$, $x = b$ and $y = \mathrm{f}(x)$ (you may want to sketch this!). Let the mass per unit area of the lamina be ρ.

You can think of this lamina as being made up of small rectangles of width δx. The area of one of these is $y\delta x$, and the centre of mass will by at $\left(x, \frac{1}{2}y\right)$ (this is assuming that δx is small!). Again, draw a sketch to help convince you of this.

We therefore have a set of small rectangles of mass $\rho y \delta x$ and centre of mass $\left(x, \frac{1}{2}y\right)$. Using the weighted mean approach for a composite lamina we have:

$$\bar{x} = \frac{\displaystyle\sum_{x=a}^{b} (\rho y \delta x) x}{\displaystyle\sum_{x=a}^{b} \rho y \delta x} \qquad \text{and} \qquad \bar{y} = \frac{\displaystyle\sum_{x=a}^{b} (\rho y \delta x) \tfrac{1}{2} y}{\displaystyle\sum_{x=a}^{b} \rho y \delta x}$$

Cancelling the ρ's and taking the limit as $\delta x \to 0$ we have:

$$\bar{x} = \frac{\displaystyle\int_{x=a}^{b} xy \,\mathrm{d}x}{\displaystyle\int_{x=a}^{b} y \,\mathrm{d}x} \qquad \text{and} \qquad \bar{y} = \frac{\displaystyle\int_{x=a}^{b} \tfrac{1}{2} y^2 \,\mathrm{d}x}{\displaystyle\int_{x=a}^{b} y \,\mathrm{d}x}$$

Volume of revolution

Consider a solid shape which is formed by rotating the region enclosed by the line $y = \mathrm{f}(x)$, the x-axis and the line $x = a$ and $x = b$ around the x-axis.

Firstly the x-axis will be a plane of symmetry for the shape made in this way, so the centre of mass will lie on the x-axis and we only need to find the x-coordinate.

Consider the shape as being made from a series of little discs with volume $\pi y^2 \delta x$ and with centre of mass at x along the x-axis.

We then have:

$$\bar{x} = \frac{\sum \left(\rho \pi y^2 \delta x\right) x}{\sum \rho \pi y^2 \delta x}$$

and as we take the limit $\delta x \to 0$, we have:

$$\bar{x} = \frac{\int \pi y^2 x \,\mathrm{d}x}{\int \pi y^2 \,\mathrm{d}x}$$

A similar formula holds for shapes rotated about the y-axis.

Variable forces

If acceleration is given as a function of displacement, $a = \mathrm{f}(x)$, then the result $a = v\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right)$ is often useful.

The **impulse** is the change of momentum of a particle. Starting with $\mathrm{F}(t) = ma = m\dfrac{\mathrm{d}v}{\mathrm{d}t}$ we have

$$\begin{aligned}\int_{t_1}^{t_2} \mathrm{F}(t)\mathrm{d}t &= \int_U^V m\mathrm{d}v \\ &= m[v]_U^V \\ &= mU - mV\end{aligned}$$

So the impulse is given by $\displaystyle\int_{t_1}^{t_2} \mathrm{F}(t)\mathrm{d}t$, where $\mathrm{F}(t)$ is the force acting on the particle.

The **work done** by a force, $\mathrm{G}(x)$, is the change in kinetic energy. Starting with $\mathrm{G}(x) = ma = mv\dfrac{\mathrm{d}v}{\mathrm{d}x}$ we have:

$$\begin{aligned}\int_{x_1}^{x_2} \mathrm{G}(x)\mathrm{d}x &= \int_U^V mv\mathrm{d}v \\ &= m\left[\tfrac{1}{2}v^2\right]_U^V \\ &= \tfrac{1}{2}mU^2 - \tfrac{1}{2}mV^2\end{aligned}$$

So the work done is given by $\displaystyle\int_{x_1}^{x_2} \mathrm{G}(x)\,\mathrm{d}x$.

Motion in a Circle

- Angular velocity is represented by ω or $\dot{\theta}$.
- Angular speed and linear speed are related by $v = r\omega$.
- Acceleration of an object moving at constant speed around a circle is $a = r\omega^2 = \dfrac{v^2}{r}(= v\omega)$ towards the centre of the circle.
- For motion in a vertical circle the acceleration has a component of $r\omega^2 = \dfrac{v^2}{r}$ towards the centre of the circle and $r\ddot{\theta} = \dot{v}$ along the tangent to the circle.
- The result $\ddot{\theta} = \omega\dfrac{\mathrm{d}\omega}{\mathrm{d}\theta}$ can sometimes be useful.

The two most common forms of vertical circular motion are those where the particle is *constrained* to move on the circle (e.g. a bead on a wire) and those where the particle is *not constrained* (e.g. a particle on a string, or rolling inside a cylinder). In the first case the particle/bead will make complete circuits if $v > 0$ at the top. In the second case the particle needs to be moving fast enough to ensure that the string remains taut at the top of the circle or so that there is a positive reaction force between the circle and the particle.

Simple Harmonic Motion

This is the situation where a particle moves so that the acceleration is always directed towards a fixed point, O, and the acceleration is proportional to the distance from this fixed point. Examples include particles hanging from springs.

- Equation of motion $\ddot{x} = -\omega^2 x$.
- The maximum displacement from O is the amplitude, A.
- The period of the motion is $\dfrac{2\pi}{\omega}$.
- The speed is given by $v^2 = \omega^2\left(A^2 - x^2\right)$.
- The displacement is given by $x = A\sin(\omega t + \alpha)$, where α is determined by the value of x when $t = 0$.

If an additional force, $f = ma$, is applied to the system then the equation of motion becomes $\ddot{x} + \omega^2 x = a$. You may need to solve some second order differential equations.

The simple pendulum
Consider a simple pendulum which consists of a particle P with mass m attached to a light inextensible string of length l. It swings through a **small** angle each side of the equilibrium point.

We assume that the only force acting is gravity (so no air resistance etc). When the pendulum makes an angle of θ **radians** with the vertical the force can be resolved into components parallel and perpendicular to the string. The perpendicular force is the one pulling the pendulum back to the centre and is equal to $-mg\sin\theta$ (the negative sign is because it is always acting back towards the vertical).

We have $a = r\ddot{\theta}$, and $r = l$, so we have:

$$\begin{aligned} ma &= -mg\sin\theta \\ l\ddot{\theta} &= -g\sin\theta \end{aligned}$$

Then, since we have said "a small angle" (and the angle is in radians) so we have $\sin\theta \approx \theta$ and we end up with the equation $\ddot{\theta} = -\frac{g}{l}\theta$. Therefore the motion of the pendulum is (approximately) simple harmonic motion with a period of $2\pi\sqrt{\frac{l}{g}}$. Note that the period does not depend on the mass of the pendulum!

Damped Harmonic Motion

Usually there is a force resisting the motion of the particle. This could be friction, air resistance or other factors. Assuming that the damping force is proportional to the velocity of the particle, and acts in the opposite direction of the movement of the particle we have:

$$m\ddot{x} = -kx - c\dot{x} \implies m\ddot{x} + c\dot{x} + kx = 0$$

See the STEP 3 Differential equations module for methods to solve second order differential equations.

Relative Motion

If the position vector of particle A is $\mathbf{a}$ and the position vector of particle B is $\mathbf{b}$ then the position of A relative to B is given by $\mathbf{x}_R = \mathbf{a} - \mathbf{b}$. The two particles will collide when $\mathbf{a} - \mathbf{b} = 0$.

The relative velocity of particle A with respect to particle B is $\mathbf{v}_R = \dot{\mathbf{a}} - \dot{\mathbf{b}}$. The particles will collide if $\mathbf{v}_R = k\overrightarrow{A_0B_0}$ (where $k > 0$ and A_0, B_0 are the initial positions of A and B).

If the particles do not collide, you can find when they are closest by when finding when $|\mathbf{x}_R|$ is minimised. This can be done by differentiating with respect to t (might be easiest to consider $|\mathbf{x}_R|^2$ in this case) or by finding when $\mathbf{x}_R \cdot \mathbf{v}_R = 0$.