1 2009 S2 Q9

(i) A uniform lamina $OXYZ$ is in the shape of the trapezium shown in the diagram. It is right-angled at $O$ and $Z$, and $OX$ is parallel to $YZ$. The lengths of the sides are given by $OX = 9$ cm, $XY = 41$ cm, $YZ = 18$ cm and $ZO = 40$ cm. Show that its centre of mass is a distance 7 cm from the edge $OZ$.

(ii) The diagram shows a tank with no lid made of thin sheet metal. The base $OXUT$, the back $OTWZ$ and the front $XUVY$ are rectangular, and each end is a trapezium as in part (i). The width of the tank is $d$ cm.

Show that the centre of mass of the tank, when empty, is a distance

$$\frac{3(140 + 11d)}{5(12 + d)}$$ cm

from the back of the tank.

The tank is then filled with a liquid. The mass per unit volume of this liquid is $k$ times the mass per unit area of the sheet metal. In the case $d = 20$, find an expression for the distance of the centre of mass of the filled tank from the back of the tank.
2  2010 S2 Q11
A uniform rod $AB$ of length $4L$ and weight $W$ is inclined at an angle $\theta$ to the horizontal. Its lower end $A$ rests on a fixed support and the rod is held in equilibrium by a string attached to the rod at a point $C$ which is $3L$ from $A$. The reaction of the support on the rod acts in a direction $\alpha$ to $AC$ and the string is inclined at an angle $\beta$ to $CA$. Show that

$$\cot \alpha = 3 \tan \theta + 2 \cot \beta.$$  

Given that $\theta = 30^\circ$ and $\beta = 45^\circ$, show that $\alpha = 15^\circ$.

3  2007 S2 Q11
*In this question take the acceleration due to gravity to be $10 \text{ m s}^{-2}$ and neglect air resistance.*

The point $O$ lies in a horizontal field. The point $B$ lies $50 \text{ m}$ east of $O$. A particle is projected from $B$ at speed $25 \text{ m s}^{-1}$ at an angle $\arctan \frac{1}{2}$ above the horizontal and in a direction that makes an angle $60^\circ$ with $OB$; it passes to the north of $O$.

(i) Taking unit vectors $i$, $j$ and $k$ in the directions east, north and vertically upwards, respectively, find the position vector of the particle relative to $O$ at time $t$ seconds after the particle was projected, and show that its distance from $O$ is

$$5(t^2 - \sqrt{5}t + 10) \text{ m}.$$  

When this distance is shortest, the particle is at point $P$. Find the position vector of $P$ and its horizontal bearing from $O$.

(ii) Show that the particle reaches its maximum height at $P$.

(iii) When the particle is at $P$, a marksman fires a bullet from $O$ directly at $P$. The initial speed of the bullet is $350 \text{ m s}^{-1}$. Ignoring the effect of gravity on the bullet show that, when it passes through $P$, the distance between $P$ and the particle is approximately $3 \text{ m}$.
4  **2011 S2 Q9**

Two particles, $A$ of mass $2m$ and $B$ of mass $m$, are moving towards each other in a straight line on a smooth horizontal plane, with speeds $2u$ and $u$ respectively. They collide directly. Given that the coefficient of restitution between the particles is $e$, where $0 < e \leq 1$, determine the speeds of the particles after the collision.

After the collision, $B$ collides directly with a smooth vertical wall, rebounding and then colliding directly with $A$ for a second time. The coefficient of restitution between $B$ and the wall is $f$, where $0 < f \leq 1$. Show that the velocity of $B$ after its second collision with $A$ is

$$\frac{2}{3}(1 - e^2)u - \frac{1}{3}(1 - 4e^2)fu$$

towards the wall and that $B$ moves towards (not away from) the wall for all values of $e$ and $f$.

5  **2008 S1 Q9**

This was an old STEP I question which includes content from the new (2019 onwards) STEP II specification.

Two identical particles $P$ and $Q$, each of mass $m$, are attached to the ends of a diameter of a light thin circular hoop of radius $a$. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, $P$ is in contact with the table. At time $t$, the hoop has rotated through an angle $\theta$. Write down the position at time $t$ of $P$, relative to its starting point, in cartesian coordinates, and determine its speed in terms of $a$, $\theta$ and $\dot{\theta}$. Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.

Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.
6 2000 S1 Q10

This was an old STEP I question which includes content from the new (2019 onwards) STEP II specification.

Three particles $P_1$, $P_2$ and $P_3$ of masses $m_1$, $m_2$ and $m_3$ respectively lie at rest in a straight line on a smooth horizontal table. $P_1$ is projected with speed $v$ towards $P_2$ and brought to rest by the collision. After $P_2$ collides with $P_3$, the latter moves forward with speed $v$. The coefficients of restitution in the first and second collisions are $e$ and $e'$, respectively. Show that

$$e' = \frac{m_2 + m_3 - m_1}{m_1}.$$  

Show that $2m_1 \geq m_2 + m_3 \geq m_1$ for such collisions to be possible.

If $m_1$, $m_3$ and $v$ are fixed, find, in terms of $m_1$, $m_3$ and $v$, the largest and smallest possible values for the final energy of the system.

It is important to think of a good way of naming the velocities of the particles before and after each collision; this of course must be carefully set out in your answer.

Don’t forget, when you are trying to derive the inequalities, that $0 \leq e \leq 1$.

7 2008 S3 Q10

This is an old STEP III question, which is now on the STEP II specification

A long string consists of $n$ short light strings joined together, each of natural length $\ell$ and modulus of elasticity $\lambda$. It hangs vertically at rest, suspended from one end. Each of the short strings has a particle of mass $m$ attached to its lower end. The short strings are numbered 1 to $n$, the $n$th short string being at the top. By considering the tension in the $r$th short string, determine the length of the long string. Find also the elastic energy stored in the long string.

A uniform heavy rope of mass $M$ and natural length $L_0$ has modulus of elasticity $\lambda$. The rope hangs vertically at rest, suspended from one end. Show that the length, $L$, of the rope is given by

$$L = L_0 \left(1 + \frac{Mg}{2\lambda}\right),$$

and find an expression in terms of $L$, $L_0$ and $\lambda$ for the elastic energy stored in the rope.