STEP Support Programme

STEP 2 Miscellaneous Questions: Hints

1 Don't forget to do the first thing requested in the "stem"! Remember that x, y, z and k are all positive, so are all greater than zero.

(i) You can eliminate y and divide by k (as long as you state why $k \neq 0$). A perfect square looks like (something)^2 and is greater than or equal to zero.

If $k = 20$ you can use the bounds on z^3 to find some possible values for z. Only one of these will make $\left(4z^3 - k^2\right)/3$ a perfect square.

(ii) Follow an almost identical method here as in part **(ii)**. Start by eliminating y and then consider $\left(4kz - z^4\right)/3$.

2 Life will be easier if you define a coordinate system for the tetrahedron. I would suggest having AB on the x-axis and C on the y-axis. You can then find coordinates for A, B, C.

Throughout this question it will be helpful to sketch some triangles. There will be lots of opportunities to use Pythagoras' Theorem.

(i) By symmetry, P will be where the three medians of the triangle ABC meet. This point is called the centroid can can be found by finding the mean of the coordinates of the three vertices, or by using the fact that the centroid is two thirds of the way down the median from a vertex.

(ii) You could find the cosine of the angle between faces ABC and ABD.

(iii) The largest sphere will touch every face of the tetrahedron, and the radius will be perpendicular to the faces. Sketch a triangle showing points D, P and the midpoint of AB.

3 If interchanging x and y has no effect then the curve is symmetrical in $y = x$, and if you replace x with $-x$ this has no effect then the curve is symmetrical in the y-axis. Finding the symmetries helps you to sketch the curves. You can pick values of u and v to help you sketch (just as long as the two curves intersect, and don't mark numbers on the axes etc.).

There should be two intersections in the first quadrant and two in the third quadrant. The conditions on α and β identify which one is A.

You can show that the lengths AB and CD are the same and lengths BC and AD are the same, but this is not sufficient as $ABCD$ could be a parallelogram. You will need to show that at least one pair of lines are perpendicular (probably more pairs depending on your method).

Find the area of $ABCD$ in terms of α and β first, and use the equations of the curves to find relationships for u and v in terms of α and β. The very last part is intended to be a check that your expression for the area is correct.

4 Since $\mathrm{p}(x) - 1$ is divisible by $(x-1)^5$ you can write $\mathrm{p}(x) - 1 = (x-1)^5 \times$ something. Best to use a function rather than "something" though (and you can state what sort of function it is).

(i) Substitute in $x = 1$.

(ii) Differentiate your $\mathrm{p}(x)$, and you should be able to factorise out $(x-1)^4$.

(iii) Now $\mathrm{p}(x) + 1 = (x+1)^5 \times$ something else. Follow the same steps as previously and you should be able to find what $\mathrm{p}'(x)$ is (up to a multiplicative constant). You can then integrate.

5 Don't forget to do the bit in the "stem"!

(i) First try to show that $F_i < 2F_{i-1}$. It is helpful to note that for $i \geqslant 4$ we have $F_{i-1} > F_{i-2}$. You will need the sum of a geometric progression.

For the second bit, start by showing that $\dfrac{1}{F_i} < \dfrac{1}{2F_{i-2}}$. Since this connects i to $i-2$ you will need to consider odd and even i separately.

(ii) Here do the same thing as in part **(i)**, but take more terms before using the geometric progression.

6 **(i)** If the sequence is constant then $x_{n+1} = x_n = x_1$ and $y_{n+1} = y_n = y_1$. Substitute the given values of a and b and solve the simultaneous equations to find x_1 and y_1.

(ii) Here we need $x_1 = x_3 = -1$ and $y_1 = y_3 = 1$. Find x_3 and y_3 in terms of a and b and hence solve. Remember that you want **period 2**, so (x_2, y_2) needs to be different!

7 **(i)** Remember that the binomial expansion:

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

where k is a rational number, only holds when $|x| < 1$. Hence we need $\left|\frac{k}{100}\right| < 1$.

(a) Note that:

$$\begin{aligned}\left(1 + \frac{8}{100}\right)^{\frac{1}{2}} &= \sqrt{\left(\frac{108}{100}\right)} \\ &= \sqrt{\left(\frac{36 \times 3}{100}\right)} \\ &= \frac{6}{10} \times \sqrt{3}\end{aligned}$$

You will have to do some decimal division!

(b) Do something similar to part **(a)**, but this time you need to find an appropriate value of k. The approximation will be better the smaller $|k|$ is.

(ii) Very similar to previous part! It might be helpful to write out some of the cube numbers.

8 Draw a big, clear diagram. You will need some circle theorems.

You should be able to write the radius of the larger circle in terms of a.

First time I did this question I had my ratio of areas the wrong way around, if this happens to you no need to panic, just try again!

Note that the expression for πR is a quadratic in q. You can find the maximum of this.

9 **(i)** Write out some of the terms long hand ($b_0 = \lambda^0 - \mu^0$ etc.). You should have two geometric series. Note that $\lambda - 1 = \sqrt{2}$.

(ii) For a "nested sum" start by evaluating the inner sum (which you have already found in part **(i)**). You can use your results from part **(i)** to help do the second sum as well, but only if you re-write the sum so that it looks like that in part **(i)**. Fortunately $b_0 = 0$ so can be added with no problems.

It is helpful to evaluate $\lambda \times \mu$.

(iii) The result from part **(i)** will simplify the first summation. For the second one, write it out longhand and simplify $(a_1 + a_3 + a_5 + \ldots + a_{2n+1} = \lambda^1 + \mu^1 + \ldots)$.

Some of the work in part **(ii)** will be useful to refer to.