# STEP Support Programme

# Pure STEP 3 Solutions

## 2012 S3 Q6

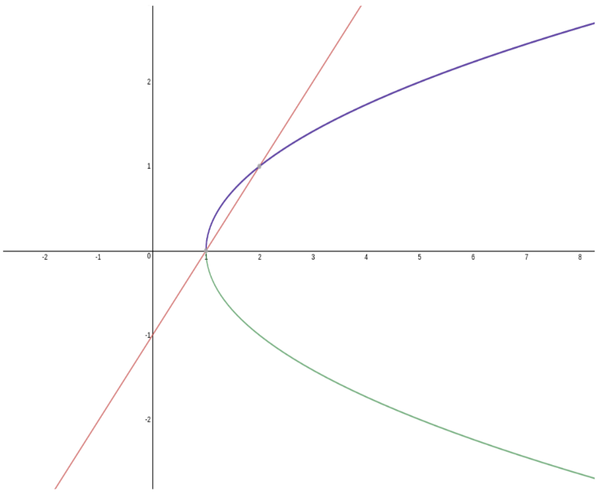
### 1 Preparation

**(i)** Completing the square on $x$ gives $(x+1)^2 + y^2 = 1$, so the centre is at $(-1, 0)$ and the radius is 1.

**(ii)** First draw a sketch of $y = x^4 - x = x(x^3 - 1)$. This has roots at $x = 0$ and $x = 1$, and you can show that that there is a minimum point when $x = \sqrt[3]{\frac{1}{4}}$. Using the graph we have $x^4 - x \geqslant 0$ when $x \leqslant 0$ or $x \geqslant 1$.

For the second part, sketch $y = -x^2 - \frac{1}{x}$. This crosses the $x$-axis at $x = -1$ **only**, is asymptotic to the $y$-axis and has a stationary point at $x = \sqrt[2]{\frac{1}{2}}$ (when $y$ is negative). Using the sketch we have $-x^2 - \frac{1}{x} \geqslant 0$ when $-1 \leqslant x < 0$.

**(iii)** You should have a diagram looking something like the one below:




Note that $y^2 = x - 1$ includes both $y = \sqrt{x-1}$ and $y = -\sqrt{x-1}$. You can find some more on sketching graphs of $y^2 = \mathrm{f}(x)$ in the STEP II Curve Sketching module. For the second and third graphs the gradient tends to infinity as $x$ tends to 1 (from above).

## 2 The STEP III question

**(i)** Substituting in $z = x + \mathrm{i}y$ gives:

$$(x+\mathrm{i}y)^2 + p(x+\mathrm{i}y) + 1 = 0$$
$$x^2 - y^2 + 2\mathrm{i}xy + px + p\mathrm{i}y + 1 = 0$$

Real Part: $$x^2 - y^2 + px + 1 = 0$$

Imaginary Part: $$2xy + py = (2x+p)y = 0$$

Taking the imaginary part then either $y = 0$ or $p = -2x$. Taking $y = 0$ in the real part gives:

$$x^2 - px + 1 = 0 \quad \Longrightarrow \quad p = -\frac{x^2+1}{x} \quad \text{for } x \neq 0$$

Therefore we either have $p = -2x$ or $p = -\dfrac{x^2+1}{x}$.

If $p = -2x$ then the real part equation becomes $x^2 - y^2 - 2x^2 + 1 = 0$ which becomes $x^2 + y^2 = 1$, i.e. a circle radius 1 centre $(0,0)$.

If $p = -\dfrac{x^2+1}{x}$ then we have $y = 0$, which is the $x$-axis with the point $(0,0)$ excluded since we cannot have $x = 0$.

**(ii)** Here we have:

$$p(x+\mathrm{i}y)^2 + (x+\mathrm{i}y) + 1 = 0$$
$$px^2 - py^2 + 2p\mathrm{i}xy + x + \mathrm{i}y + 1 = 0$$

Real Part: $$px^2 - py^2 + x + 1 = 0$$

Imaginary Part: $$2pxy + y = (2xp+1)y = 0$$

The imaginary equation gives $y = 0$ or $p = -\dfrac{1}{2x}$. Taking these in turn:

Substituting $y = 0$ into the real part equation gives $px^2 + x + 1 = 0$ i.e. $p = -\dfrac{x^2+1}{x^2}$. This means that we cannot have $x = 0$, so the locus of this case is the real $(x)$ axis with the origin excluded.

Substituting $p = -\dfrac{1}{2x}$ into the real part equation gives:

$$-\frac{1}{2x}x^2 + \frac{1}{2x}y^2 + x + 1 = 0$$
$$-x^2 + y^2 + 2x^2 + 2x = 0$$
$$x^2 + y^2 + 2x = 0$$

which is the equation of a circle centre $(-1,0)$ and radius 1. Since $p = -\frac{1}{2x}$ we need to exclude the point $x = 0$, which again is the origin.

Don't forget to sketch the loci!

**(iii)** In this last case we have:

$$\begin{aligned} & p(x+\mathrm{i}y)^2 + p^2(x+\mathrm{i}y) + 2 = 0 \\ & px^2 - py^2 + 2p\mathrm{i}xy + p^2x + p^2\mathrm{i}y + 2 = 0 \\ \text{Real Part:} \quad & px^2 - py^2 + p^2x + 2 = 0 \\ \text{Imaginary Part:} \quad & 2pxy + p^2y = (2x+p)py = 0 \end{aligned}$$

The first thing to note is that the real part equation implies that we cannot have $p = 0$. Hence the imaginary equation implies that either $y = 0$ or $p = -2x$.

If $y = 0$ then the real part equation becomes:

$$px^2 + p^2x + 2 = 0$$

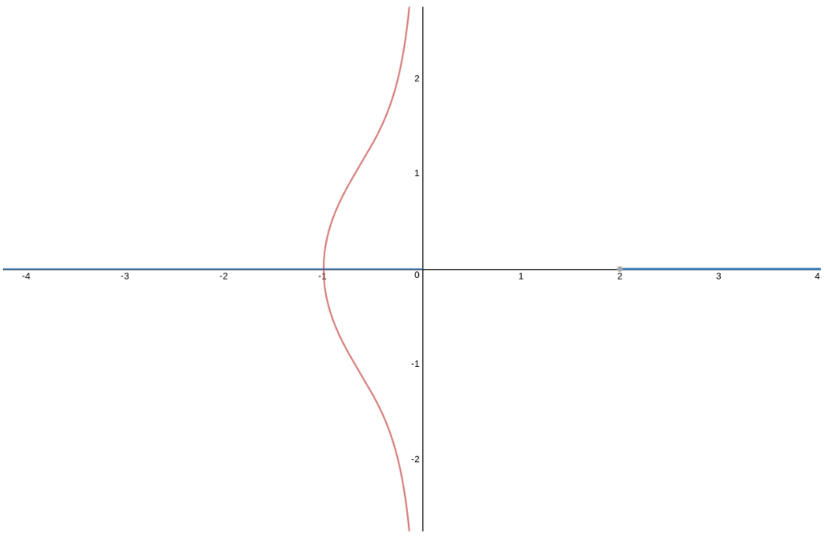
Solving for $p$ gives $p = \dfrac{-x^2 \pm \sqrt{x^4 - 8x}}{2x}$. $p$ is real if and only if $x^4 - 8x \geqslant 0$ and $x \neq 0$, i.e. iff $x < 0$ or $x \geqslant 2$. This locus is the real $(x)$ axis with the segment $0 \leqslant x < 2$ excluded.

If $p = -2x$ then we have:

$$-2x^3 + 2xy^2 + 4x^3 + 2 = 0 \quad \Longrightarrow \quad x^3 + xy^2 + 1 = 0$$

This gives $y^2 = -\frac{x^3+1}{x}$.

The picture looks something like this:




To make it clearer, you could put an empty circle on the origin to show that this pint is excluded.

# 2001 S3 Q1

**3 Preparation**

**(i)**

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big(\sin(\ln x)\big) &= \cos(\ln x) \times \frac{1}{x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{e}^{2\sin(\ln x)}\big) &= 2\cos(\ln x) \times \frac{1}{x} \times \mathrm{e}^{2\sin(\ln x)} \\
&= \frac{2}{x}\cos(\ln x)\mathrm{e}^{2\sin(\ln x)}
\end{aligned}$$

**(ii)** **Base case** When $n = 1$ the conjecture becomes $\frac{\mathrm{d}}{\mathrm{d}x}x = 1$, which is true (sketch $y = x$ if you must!)

**Induction step** Assume the conjecture is true when $n = k$, i.e. we have $\frac{\mathrm{d}}{\mathrm{d}x}x^k = kx^{k-1}$. Now consider the case $n = k + 1$. We have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}x^{k+1} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x \times x^k\right) \\
&= x \times \frac{\mathrm{d}}{\mathrm{d}x}x^k + x^k \times \frac{\mathrm{d}}{\mathrm{d}x}x \\
&= x \times kx^{k-1} + x^k \times 1 \\
&= kx^k + x^k \\
&= (k+1)x^k
\end{aligned}$$

Which is the result when $n = k + 1$. Hence if it is true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$ it is true for all integers $n \geqslant 1$.

**(iii)** If $\mathrm{f}(x) = \sin x$ then we have:

$$\begin{aligned}
\mathrm{f}'(x) &= \phantom{-}\cos(x) &\implies\quad \mathrm{f}'(0) &= \phantom{-}1 \\
\mathrm{f}''(x) &= -\sin(x) &\implies\quad \mathrm{f}''(0) &= \phantom{-}0 \\
\mathrm{f}^{(3)}(x) &= -\cos(x) &\implies\quad \mathrm{f}^{(3)}(0) &= -1 \\
\mathrm{f}^{(4)}(x) &= \phantom{-}\sin(x) &\implies\quad \mathrm{f}^{(4)}(0) &= \phantom{-}0 \\
\mathrm{f}^{(5)}(x) &= \phantom{-}\cos(x) &\implies\quad \mathrm{f}^{(5)}(0) &= \phantom{-}1
\end{aligned}$$

This means that the first three non-zero terms in the Maclaurin's expansion of $\sin x$ are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Hence:

$$\begin{aligned}\sin 0.1 &\approx 0.1 - \frac{0.1^3}{3!} + \frac{0.1^5}{5!} \\ &= 0.1 - \frac{1}{6} \times \frac{1}{1000} + \frac{1}{12} \times \frac{1}{1,000,000} \\ &= 0.1 - 0.00016666\cdots + 0.000000083333\cdots \\ &= 0.099833333333\cdots + 0.000000083333\cdots \\ &= 0.099833416666\cdots\end{aligned}$$

So to seven decimal places we have $\sin 0.1 = 0.0998334$.

## 4 The STEP III question

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x+\sqrt{x^2+1}\right)\right) &= \frac{1}{x+\sqrt{x^2+1}} \times \left(1+\tfrac{1}{2}\times 2x \times \left(x^2+1\right)^{-\frac{1}{2}}\right)\\ &= \frac{1}{x+\sqrt{x^2+1}} \times \left(1+\frac{x}{\sqrt{x^2+1}}\right)\\ &= \frac{1}{x+\sqrt{x^2+1}} \times \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\\ &= \frac{1}{\sqrt{x^2+1}} \quad \text{as required.}\end{aligned}$$

Since we are going to need the second derivative as well for the base case, lets work it out here to give:

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\tfrac{1}{2}\times 2x \times \left(x^2+1\right)^{-\frac{3}{2}}\\ &= -\frac{x}{\left(x^2+1\right)^{3/2}}\end{aligned}$$

**Base case** When $n=0$ we have:

$$\begin{aligned}&\left(x^2+1\right)y^{(2)}+(2\times 0+1)xy^{(1)}+0^2y\\ =&\left(x^2+1\right)\times -\frac{x}{\left(x^2+1\right)^{3/2}}+x\times\frac{1}{\sqrt{x^2+1}}\\ =&-\frac{x\left(x^2+1\right)}{\left(x^2+1\right)^{3/2}}+\frac{x}{\sqrt{x^2+1}}\\ =&\,0\end{aligned}$$

Hence the conjecture is true when $n=0$. Assume now that the conjecture is true when $n=k$, so we have:

$$\left(x^2+1\right)y^{(k+2)}+(2k+1)xy^{(k+1)}+k^2y^{(k)}=0$$

Differentiating this with respect to $x$ gives:

$$\begin{aligned}\left[\left(x^2+1\right)y^{(k+3)}+2xy^{(k+2)}\right]+\left[(2k+1)xy^{(k+2)}+(2k+1)y^{(k+1)}\right]+k^2y^{(k+1)} &= 0\\ \left(x^2+1\right)y^{(k+3)}+\Big[2x+(2k+1)x\Big]y^{(k+2)}+\Big[(2k+1)+k^2\Big]y^{(k+1)} &= 0\\ \left(x^2+1\right)y^{(k+3)}+\Big[2+2k+1\Big]xy^{(k+2)}+\Big[k^2+2k+1\Big]y^{(k+1)} &= 0\\ \left(x^2+1\right)y^{(k+3)}+\Big[2(k+1)+1\Big]xy^{(k+2)}+\Big[(k+1)^2\Big]y^{(k+1)} &= 0\end{aligned}$$

Which is the conjecture with $n=k+1$. Hence if it is true for $n=k$ then it is true for $n=k+1$ and as it is true for $n=0$ it is true for all integers $n\geqslant 0$.

This result can be rearranged to give:

$$(x^2+1)y^{(n+2)} = -(2n+1)xy^{(n+1)} - n^2y^{(n)}$$

and so, if $x=0$ we have:

$$y^{(n+2)} = -n^2y^{(n)}\,.$$

We now have:

$$\begin{aligned}
y(0) &= \ln 1 = 0\\
y'(0) &= \frac{1}{\sqrt{0^2+1}} = 1\\
y^{(2)}(0) &= -\frac{0}{(0^2+1)^{3/2}} = 0\\
y^{(3)}(0) &= -1^2y^{(1)}(0) = -1\\
y^{(4)}(0) &= -2^2y^{(2)}(0) = 0\\
y^{(5)}(0) &= -3^2y^{(3)}(0) = 3^2\\
y^{(6)}(0) &= -4^2y^{(4)}(0) = 0\\
y^{(7)}(0) &= -5^2y^{(5)}(0) = -5^2\times 3^2
\end{aligned}$$

Hence the Maclaurin's expansion is:

$$\begin{aligned}
y(x) &= y(0) + xy'(0) + \frac{x^2}{2!}y^{(2)}(0) + \frac{x^3}{3!}y^{(3)}(0) + \ldots\\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!}\times 3^2 - \frac{x^7}{7!}\times\left(3^2\times 5^2\right)\\
&= x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}
\end{aligned}$$

## 2009 S3 Q8

**5 Preparation**

**(i)** There are different ways you can approach these. Below is one possible set of solutions.

**(a)** Limit is 0, by **(iv)**. Note that the limits if $\mathrm{f}(x)$ and $\mathrm{g}(x)$ must be *finite*.

**(b)** Limit is 5, by **(ii)** and **(i)**.

**(c)** $\cos x$ is continuous, so using **(v)**: $\lim_{t\to\infty}\left(\cos\left(\frac{1}{t}\right)\right) = \cos\left(\lim_{t\to\infty}\frac{1}{t}\right) = \cos(0) = 1$.

**(d)** $\mathrm{e}^x$ is continuous, so we can use **(v)** to get a limit of 1.

**(ii)** If we consider $\dfrac{t}{\mathrm{e}^t}$, both the numerator and denominator tend to infinity as $t \to \infty$, so we cannot deduce anything about the limit like this. Instead, try:

$$\begin{aligned}\mathrm{e}^{-t}t &= \frac{t}{\mathrm{e}^t} \\ &= \frac{t}{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots} \\ &= \frac{1}{\frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \dots}\end{aligned}$$

Now as $t \to \infty$, The numerator stays as 1, but the denominator tends to infinity hence $\lim_{t\to\infty} \mathrm{e}^{-t}t = 0$.

A useful rule of thumb is that "exponentials always beat polynomials (eventually)".

**(iii)** If $x = \mathrm{e}^{-t}$ then as $t \to \infty$, $x \to 0$. We have:

$$\begin{aligned}\lim_{x\to 0}(x\ln x) &= \lim_{t\to\infty}\left(\mathrm{e}^{-t}\ln\left(\mathrm{e}^{-t}\right)\right) \\ &= \lim_{t\to\infty}\left(-t\mathrm{e}^{-t}\right) \\ &= 0 \qquad \text{by part } \textbf{(ii)}.\end{aligned}$$

**(iv)** **(a)**

$$\begin{aligned}\int_r^1 \ln x \,\mathrm{d}x &= \int_r^1 1 \times \ln x \,\mathrm{d}x \\ &= \Big[x \times \ln x\Big]_r^1 - \int_r^1 x \times \frac{1}{x}\,\mathrm{d}x \\ &= r\ln r - 0 - \int_r^1 1\,\mathrm{d}x \\ &= r\ln r - \Big[x\Big]_r^1 \\ &= r\ln r - (1 - r)\end{aligned}$$

**(b)** Taking the limit as $r \to 0$, and using the result from part **(ii)**, we have:

$$\int_0^1 \ln x\,\mathrm{d}x = -1\,.$$

## 6 The STEP III question

Don't forget to do the first bit! This is called the "Stem" of the question and anything proved or given here is very likely to be useful in the rest of the question.

Setting $x = \mathrm{e}^{-t}$ gives us:

$$\begin{aligned}\lim_{x\to 0} x^m (\ln x)^n &= \lim_{t\to\infty} \mathrm{e}^{-mt} (-t)^n \\ &= (-1)^n \lim_{t\to\infty} \mathrm{e}^{-mt} t^n \\ &= 0 \qquad \text{using the given result.}\end{aligned}$$

**(i)**

$$\begin{aligned}\lim_{x\to 0} x^x &= \lim_{x\to 0} \mathrm{e}^{x\ln x} \\ &= \mathrm{e}^{\lim_{x\to 0}(x\ln x)} \\ &= \mathrm{e}^0 \\ &= 1\end{aligned}$$

**(ii)**

$$\begin{aligned}I_{n+1} &= \int_0^1 x^m (\ln x)^{n+1} \, \mathrm{d}x \\ &= \left[\frac{x^{m+1}}{(m+1)}(\ln x)^{n+1}\right]_0^1 - \int_0^1 \frac{x^{m+1}}{(m+1)} \times (n+1)(\ln x)^n \frac{1}{x} \, \mathrm{d}x \\ &= 0 - \frac{n+1}{m+1}\int_0^1 x^m (\ln x)^n \, \mathrm{d}x \\ &= -\frac{n+1}{m+1} I_n\end{aligned}$$

Note that the third line used the result from the "Stem" of the question.

Now we have:

$$\begin{aligned}I_n &= -\frac{n}{m+1} I_{n-1} \\ &= -\frac{n}{m+1} \times -\frac{n-1}{m+1} I_{n-2} \\ &= -\frac{n}{m+1} \times -\frac{n-1}{m+1} \times -\frac{n-2}{m+1} \times \ldots \times -\frac{1}{m+1} I_0 \\ &= (-1)^n \frac{n!}{(m+1)^n} I_0\end{aligned}$$

Since $I_0 = \displaystyle\int_0^1 x^m \, \mathrm{d}x = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}$, we have:

$$I_n = (-1)^n \frac{n!}{(m+1)^{n+1}} \, .$$

**(iii)**

$$\begin{aligned}
\int_0^1 x^x \,\mathrm{d}x &= \int_0^1 \mathrm{e}^{x\ln x}\,\mathrm{d}x \\
&= \int_0^1 \left(1 + (x\ln x) + \frac{(x\ln x)^2}{2!} + \frac{(x\ln x)^3}{3!} + \dots\right)\,\mathrm{d}x \\
&= [x]_0^1 + \int_0^1 x\ln x\,\mathrm{d}x + \frac{1}{2!}\int_0^1 x^2(\ln x)^2\,\mathrm{d}x + \frac{1}{3!}\int_0^1 x^3(\ln x)^3\,\mathrm{d}x + \dots \\
&= 1 + (-1)^1\frac{1!}{(1+1)^{1+1}} + \frac{1}{\cancel{2!}} \times (-1)^2\frac{\cancel{2!}}{(2+1)^{2+1}} + \frac{1}{\cancel{3!}} \times (-1)^3\frac{\cancel{3!}}{(3+1)^{3+1}} + \dots \\
&= 1 - \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^3 - \left(\frac{1}{4}\right)^4 + \dots
\end{aligned}$$

## 2006 S3 Q7

**7 Preparation**

**(i)**

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh x) = \tfrac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right) = \sinh x\,.$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh x) = \tfrac{1}{2}\left(\mathrm{e}^{x} - (-\mathrm{e}^{-x})\right) = \cosh x\,.$$

**(ii)** $\cosh x$ is an even function (i.e. $\cosh(-x) = \cosh x$) and as such it is symmetrical about the $y$-axis. The minimum value is when $x = 0, y = 1$, and it does not cross the $x$-axis.

$\sinh x$ is an odd function, and so it had rotational symmetry about the origin. It crosses the axes at $(0, 0)$ ad the gradient is always positive.

Try using Desmos to sketch the graphs.

**(iii)**

$$\cosh x + \sinh x = \tfrac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) + \tfrac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right) = \mathrm{e}^{x}$$
$$\cosh x - \sinh x = \tfrac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right) - \tfrac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right) = \mathrm{e}^{-x}$$

**(iv)**

$$\begin{aligned}(\cosh x)^2 - (\sinh x)^2 &= \left(\tfrac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)\right)^2 - \left(\tfrac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right)\right)^2 \\ &= \tfrac{1}{4}\left(\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}\right) - \tfrac{1}{4}\left(\mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}\right) \\ &= 1\end{aligned}$$

**(v)** $y = \cosh x + c$, and $y = 2$ when $x = 0$ implies that $c = 1$ and so $y = \cosh x + 1$.

**(vi)**

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sinh y}{\cosh y} \\ \int \frac{\cosh y}{\sinh y}\,\mathrm{d}y &= \int 1\,\mathrm{d}x \\ \ln|\sinh y| &= x + c \\ \sinh y &= A\mathrm{e}^{x}\end{aligned}$$

## 8 The STEP III question

**(i)** Using the quadratic formula gives:

$$\begin{aligned} u &= \frac{-2\sinh x \pm \sqrt{4\sinh^2 x + 4}}{2} \\ &= -\sinh x \pm \cosh x \\ &= -\mathrm{e}^x \quad \text{or} \quad \mathrm{e}^{-x} \end{aligned}$$

Taking each of these solutions in turn gives:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= -\mathrm{e}^x & \text{or} && \frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e}^{-x} \\ y &= -\mathrm{e}^x + a & \text{or} && y &= -\mathrm{e}^{-x} + b \end{aligned}$$

Since we want $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ (which excludes $\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^x$) at $x = 0$, we have $y = -\mathrm{e}^{-x} + 1$.

**(ii)** Using the quadratic formula gives:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-2 \pm \sqrt{4 + 4\sinh^2 y}}{2\sinh y} \\ &= \frac{1 \pm \cosh y}{\sinh y} \end{aligned}$$

Hence we have:

$$\begin{aligned} \pm \int \frac{\sinh y}{\cosh y \mp 1}\,\mathrm{d}y &= \int 1\,\mathrm{d}x \\ \pm \ln(\cosh y \mp 1) &= x + c \end{aligned}$$

We can only set $y = 0$ into $-\ln(\cosh y + 1)$, so doing this gives:

$$-\ln 2 = 0 + c \quad \Longrightarrow \quad -\ln(\cosh y + 1) = x - \ln 2$$

and so:

$$\begin{aligned} \ln(\cosh y + 1) &= \ln 2 - x \\ \cosh y + 1 &= 2\mathrm{e}^{-x} \\ \cosh y &= 2\mathrm{e}^{-x} - 1 \end{aligned}$$

Since $\cosh y \geqslant 1$, $y$ is only defined for $x \leqslant 0$. The asymptotes are what happens as $x \to -\infty$ when $y \to \pm\infty$.

$$\begin{aligned} y \to +\infty &\quad\Longrightarrow\quad \tfrac{1}{2}\mathrm{e}^{y} \approx 2\mathrm{e}^{-x} &\quad\Longrightarrow\quad y \approx \ln 4 - x \\ y \to -\infty &\quad\Longrightarrow\quad \tfrac{1}{2}\mathrm{e}^{-y} \approx 2\mathrm{e}^{-x} &\quad\Longrightarrow\quad y \approx -\ln 4 + x \end{aligned}$$

Therefore the asymptotes are $y = \pm(-x + \ln 4)$.

# 1988 S2 Q5

## 9 Preparation

**(i) (a) Base case**

When $n = 1$ we have $(\cos\theta + \mathrm{i}\sin\theta)^1 = \cos\theta + \mathrm{i}\sin\theta$, which is true.
You could instead show it is true when $n = 0$.

**Inductive step**

Assume the conjecture is true when $n = k$, so $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + \mathrm{i}\sin k\theta$.
Now consider $n = k + 1$:

$$\begin{aligned}(\cos\theta + \mathrm{i}\sin\theta)^{k+1} &= (\cos\theta + \mathrm{i}\sin\theta)^k \times (\cos\theta + \mathrm{i}\sin\theta)\\ &= (\cos k\theta + \mathrm{i}\sin k\theta) \times (\cos\theta + \mathrm{i}\sin\theta)\\ &= \cos k\theta\cos\theta + \mathrm{i}\cos k\theta\sin\theta + \mathrm{i}\sin k\theta\cos\theta - \sin k\theta\sin\theta\\ &= \cos k\theta\cos\theta - \sin k\theta\sin\theta + \mathrm{i}\left(\cos k\theta\sin\theta + \sin k\theta\cos\theta\right)\\ &= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta\end{aligned}$$

Therefore if it is true when $n = k$, it is true when $n = k + 1$, and since it is true when $n = 1$ it is true for all integers $n \geqslant 1$.

**(b)**

$$\begin{aligned}z^5 &= 1\\ r^5\left(\cos\theta + \mathrm{i}\sin\theta\right)^5 &= 1\\ r^5\left(\cos 5\theta + \mathrm{i}\sin 5\theta\right) &= 1\end{aligned}$$

Then equating real parts gives $r^5\cos 5\theta = 1$ (hence $r \neq 0$) and equating imaginary parts gives $r^5 \sin 5\theta = 0 \implies \sin 5\theta = 0$. This means that $\cos 5\theta = \pm 1$.

Since $r \geqslant 0$, we must have $r = 1$ and $\cos 5\theta = 1 \implies \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}$ and $\frac{8\pi}{5}$. We therefore have $z = \cos\left(\frac{2k\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2k\pi}{5}\right)$ for $k = 0, 1, 2, 3, 4$. These are the 5 *fifth roots of unity*.

**(c)** We have:

$$\begin{aligned}\cos 5\theta + \mathrm{i}\sin 5\theta &= (\cos\theta + \mathrm{i}\sin\theta)^5\\ &= (\cos\theta)^5 + 5 \times (\cos\theta)^4\,(\mathrm{i}\sin\theta) + 10 \times (\cos\theta)^3\,(\mathrm{i}\sin\theta)^2 + \ldots\end{aligned}$$

Equating real parts gives:

$$\cos 5\theta = (\cos\theta)^5 - 10\,(\cos\theta)^3\,(\sin\theta)^2 + 5\,(\cos\theta)\,(\sin\theta)^4$$

You can then use $\sin^2\theta = 1 - \cos^2\theta$ to write this just in terms of $\cos\theta$.

**(ii)** Considering $(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n) = x^n + a_1x^{n-1} + \ldots a_{n-1}x + a_n$ gives us $\alpha_1 \times \alpha_2 \times \ldots \times \alpha_n = (-1)^n a_n$.

## 10 The STEP III question

Let $z = x + \mathrm{i}y$, so we have $(x + \mathrm{i}y)^7 = 1$. The imaginary part of this gives:

$$7x^6y - 35x^4y^3 + 21x^2y^5 - y^7 = 0$$

which means that either $y = 0$ or $7x^6 - 35x^4y^2 + 21x^2y^4 - y^6 = 0$. It we divide the second equation by $x^6$ and set $t = \dfrac{y}{x}$ (where $x \neq 0$) we get $7 - 35t^2 + 21t^4 - t^6 = 0$ i.e. $t^6 - 21t^4 + 35t^2 - 7 = 0$.

Solving $z^7 = 1$ gives $z = \mathrm{e}^{\frac{2\pi k\mathrm{i}}{7}}$ for $k = -3, -2, -1, 0, 1, 2, 3$.

Here we are taking the argument as lying between $-\pi$ and $\pi$ rather than between $0$ and $2\pi$. The reason for this should become clear in the next few lines.

When $k = 0$ we have $z = 1$, and so $y = 0$. The other 6 solutions relate to the roots of $t^6 - 21t^4 + 35t^2 - 7 = 0$.

The roots are therefore $t = \dfrac{y}{x} = \dfrac{\sin\left(\frac{2\pi k}{7}\right)}{\cos\left(\frac{2\pi k}{7}\right)} = \tan\left(\frac{2\pi k}{7}\right)$.

Using $\alpha_1 \times \alpha_2 \times \ldots \times \alpha_n = (-1)^n a_n$ gives:

$$\begin{aligned}
\tan\left(\tfrac{-6\pi}{7}\right) \times \tan\left(\tfrac{-4\pi}{7}\right) \times \tan\left(\tfrac{-2\pi}{7}\right) \times \tan\left(\tfrac{2\pi}{7}\right) \times \tan\left(\tfrac{4\pi}{7}\right) \times \tan\left(\tfrac{6\pi}{7}\right) &= (-1)^6 \times -7 \\
-\tan\left(\tfrac{6\pi}{7}\right) \times -\tan\left(\tfrac{4\pi}{7}\right) \times -\tan\left(\tfrac{2\pi}{7}\right) \times \tan\left(\tfrac{2\pi}{7}\right) \times \tan\left(\tfrac{4\pi}{7}\right) \times \tan\left(\tfrac{6\pi}{7}\right) &= -7 \\
\left(\tan\left(\tfrac{2\pi}{7}\right)\right)^2 \times \left(\tan\left(\tfrac{4\pi}{7}\right)\right)^2 \times \left(\tan\left(\tfrac{6\pi}{7}\right)\right)^2 &= 7 \\
\tan\left(\tfrac{2\pi}{7}\right) \times \tan\left(\tfrac{4\pi}{7}\right) \times \tan\left(\tfrac{6\pi}{7}\right) &= \pm\sqrt{7}
\end{aligned}$$

Then, since $\tan\left(\frac{2\pi}{7}\right)$ is positive, whilst the other two are negative we have:

$$\tan\left(\tfrac{2\pi}{7}\right) \times \tan\left(\tfrac{4\pi}{7}\right) \times \tan\left(\tfrac{6\pi}{7}\right) = \sqrt{7}$$

When $n = 9$, start by considering $z^9 = 1$, and the imaginary part of this, which gives:

$$9x^8y - \binom{9}{3}x^6y^3 + \binom{9}{5}x^4y^5 - \binom{9}{7}x^2y^7 + y^9 = 0$$

Dividing by $x^9$ and setting $t = \dfrac{y}{x}$ gives either $y = 0$ or

$$t^8 - \binom{9}{7}t^6 + \binom{9}{5}t^4 - \binom{9}{3}t^2 + 9 = 0\,.$$

The ninth roots of unity are $z = \mathrm{e}^{\frac{2\pi k\mathrm{i}}{9}}$ for $k = -4, -3, -2, -1, 0, 1, 2, 3, 4$ where $k = 0$ corresponds to $y = 0$. We have:

$$\begin{aligned}
\tan\left(\tfrac{-8\pi}{9}\right) \times \tan\left(\tfrac{-6\pi}{9}\right) \times \ldots \times \tan\left(\tfrac{6\pi}{9}\right) \times \tan\left(\tfrac{8\pi}{9}\right) &= (-1)^8 \times 9 \\
-\tan\left(\tfrac{8\pi}{9}\right) \times -\tan\left(\tfrac{6\pi}{9}\right) \times \ldots \times \tan\left(\tfrac{6\pi}{9}\right) \times \tan\left(\tfrac{8\pi}{9}\right) &= 9 \\
\left(\tan\left(\tfrac{2\pi}{9}\right)\right)^2 \times \left(\tan\left(\tfrac{4\pi}{9}\right)\right)^2 \times \left(\tan\left(\tfrac{6\pi}{9}\right)\right)^2 \times \left(\tan\left(\tfrac{8\pi}{9}\right)\right)^2 &= 9 \\
\tan\left(\tfrac{2\pi}{9}\right) \times \tan\left(\tfrac{4\pi}{9}\right) \times \tan\left(\tfrac{6\pi}{9}\right) \times \tan\left(\tfrac{8\pi}{9}\right) &= \pm\sqrt{9}
\end{aligned}$$

Two of tan values are positive and two are negative, so we have:

$$\tan\left(\tfrac{2\pi}{9}\right) \times \tan\left(\tfrac{4\pi}{9}\right) \times \tan\left(\tfrac{6\pi}{9}\right) \times \tan\left(\tfrac{8\pi}{9}\right) = \sqrt{9}$$

For the final part, follow exactly the same idea, but you will find that $\tan\left(\frac{2\pi}{11}\right)$ and $\tan\left(\frac{4\pi}{11}\right)$ are positive whilst $\tan\left(\frac{6\pi}{11}\right)$, $\tan\left(\frac{8\pi}{11}\right)$ and $\tan\left(\frac{10\pi}{11}\right)$ are negative, so we have:

$$\tan\left(\tfrac{2\pi}{11}\right) \times \tan\left(\tfrac{4\pi}{11}\right) \times \tan\left(\tfrac{6\pi}{11}\right) \times \tan\left(\tfrac{8\pi}{11}\right) \times \tan\left(\tfrac{10\pi}{11}\right) = -\sqrt{11}\,.$$

# 2013 S3 Q3

## 11 Preparation

**(i)** This is quite hard. One way of thinking about it is to start by assuming that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{x}$. Now if we rotate everything by 120° about the origin then $\mathbf{p_1} \to \mathbf{p_2}$ etc, so that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$ is unchanged. This means that $\mathbf{x}$ stays in the same place after a rotation of 120° about the origin, therefore $\mathbf{x}$ must be at the origin and we have $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}$.

**(ii)**

$$\begin{aligned}
&\mathbf{p}_1.\mathbf{p}_1 = 1 \qquad \text{i.e. the radius of the circle}\\
&(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3).\mathbf{p}_1 = \mathbf{0}.\mathbf{p}_1 = 0\\
&(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3).\mathbf{p}_1 = 0 \implies \mathbf{p}_1.\mathbf{p}_1 + \mathbf{p}_2.\mathbf{p}_1 + \mathbf{p}_3.\mathbf{p}_1 = 0 \implies \mathbf{p}_2.\mathbf{p}_1 + \mathbf{p}_3.\mathbf{p}_1 = -1\\
&\mathbf{p}_2.\mathbf{p}_1 = \mathbf{p}_3.\mathbf{p}_1 \quad \text{(by symmetry)} \implies \mathbf{p}_2.\mathbf{p}_1 = -\tfrac{1}{2}
\end{aligned}$$

**(iii)** $(\mathbf{x} - \mathbf{p}_1).(\mathbf{x} - \mathbf{p}_1) = \mathbf{x}.\mathbf{x} - 2\mathbf{x}.\mathbf{p}_1 + \mathbf{p}_1.\mathbf{p}_1 = k^2 - 2\mathbf{x}.\mathbf{p}_1 + 1.$

$$\begin{aligned}
\sum_{i=1}^{3}(\mathbf{x} - \mathbf{p}_i).(\mathbf{x} - \mathbf{p}_i) &= \sum_{i=1}^{3}\left(k^2 - 2\mathbf{x}.\mathbf{p}_i + 1\right)\\
&= 3k^2 - 2\sum_{i=1}^{3}\mathbf{x}.\mathbf{p}_i + 3\\
&= 3k^2 - 2\mathbf{x}.\left(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3\right) + 3\\
&= 3k^2 + 3
\end{aligned}$$

**(iv)** Let $\mathbf{p}_2 = (a, b)$. Then $\mathbf{p}_2.\mathbf{p}_1 = -\frac{1}{2} \implies b = -\frac{1}{2}$. We also have $\mathbf{p}_2.\mathbf{p}_2 = 1 \implies a^2 + b^2 = 1 \implies a = \pm\frac{\sqrt{3}}{2}$. So $\mathbf{p}_2$ and $\mathbf{p}_3$ are given by $\left(\pm\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$.

## 12 The STEP III question

Similarly to before we have $(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4).\mathbf{p}_1 = 0$ and so $\mathbf{p}_2.\mathbf{p}_1+\mathbf{p}_3.\mathbf{p}_1+\mathbf{p}_4.\mathbf{p}_1 = -1$. By symmetry we have $\mathbf{p}_i.\mathbf{p}_1 = -\frac{1}{3}$ for $i = 2,3,4$ and by extending the argument we have $\mathbf{p}_i.\mathbf{p}_j = -\frac{1}{3}$.

**(i)**

$$\begin{aligned}\sum_{i=1}^{4}(XP_i)^2 &= \sum_{i=1}^{4}(\mathbf{p}_i.\mathbf{p}_i - 2\mathbf{p}_i.\mathbf{x} + \mathbf{x}.\mathbf{x}) \\ &= \sum_{i=1}^{4}(1 - 2\mathbf{p}_i.\mathbf{x} + 1) \\ &= 4 - 2\left(\sum_{i=1}^{4}\mathbf{p}_i.\mathbf{x}\right) - 4 \\ &= 8 - 2(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4).\mathbf{x} \\ &= 8 - 0 = 8\end{aligned}$$

**(ii)** $\mathbf{p}_1.\mathbf{p}_2 = -\frac{1}{3} \implies b = -\frac{1}{3}$ and $\mathbf{p}_2.\mathbf{p}_2 = 1 \implies a^2+b^2 = 1 \implies a^2 = \frac{8}{9}$ and since we are told that $a > 0$ we have $a = \dfrac{2\sqrt{2}}{3}$. Letting $P_3$ and $P_4$ be of the form $P_j = (p,q,r)$ we have:

$$\begin{aligned}\mathbf{p}_1.\mathbf{p}_j = -\frac{1}{3} &\implies r = -\frac{1}{3} \\ \mathbf{p}_2.\mathbf{p}_j = -\frac{1}{3} &\implies \frac{2\sqrt{2}}{3}p + \frac{1}{9} = -\frac{1}{3} \implies p = -\frac{4}{9}\times\frac{3}{2\sqrt{2}} = -\frac{\sqrt{2}}{3} \\ \mathbf{p}_j.\mathbf{p}_j = 1 &\implies \frac{2}{9} + q^2 + \frac{1}{9} = 1 \implies q^2 = \frac{2}{3}\end{aligned}$$

So $P_3$ and $P_4$ are given by $\left(-\dfrac{\sqrt{2}}{3}, \pm\dfrac{\sqrt{2}}{\sqrt{3}}, -\dfrac{1}{3}\right)$.

**(iii)**

$$\begin{aligned}\sum_{i=1}^{4}(XP_i)^4 &= \sum_{i=1}^{4}\left((\mathbf{p}_i-\mathbf{x}).(\mathbf{p}_i-\mathbf{x})\right)^2 \\ &= \sum_{i=1}^{4}\left(\mathbf{p}_i.\mathbf{p}_i - 2\mathbf{p}_i.\mathbf{x} + \mathbf{x}.\mathbf{x}\right)^2 \\ &= \sum_{i=1}^{4}(2 - 2\mathbf{p}_i.\mathbf{x})^2 \\ &= \sum_{i=1}^{4}4(1-\mathbf{p}_i.\mathbf{x})^2 \\ &= 4\sum_{i=1}^{4}(1-\mathbf{p}_i.\mathbf{x})^2\end{aligned}$$

Simplifying further gives:

$$\begin{aligned}\sum_{i=1}^{4}(XP_i)^4 &= 4\sum_{i=1}^{4}(1-\mathbf{p}_i.\mathbf{x})^2\\ &= 4\sum_{i=1}^{4}\left(1-2\mathbf{p}_i.\mathbf{x}+(\mathbf{p}_i.\mathbf{x})^2\right)\\ &= 4\times 4-2(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4).\mathbf{x}+4\sum_{i=1}^{4}(\mathbf{p}_i.\mathbf{x})^2\\ &= 16+4\sum_{i=1}^{4}(\mathbf{p}_i.\mathbf{x})^2\end{aligned}$$

Substituting $X=(x,y,z)$ and the values of $P_i$ found we have:

$$\begin{aligned}\sum_{i=1}^{4}(\mathbf{p}_i.\mathbf{x})^2 &= z^2+\left(\tfrac{2\sqrt{2}}{3}x-\tfrac{1}{3}z\right)^2+\left(-\tfrac{\sqrt{2}}{3}x+\tfrac{\sqrt{2}}{\sqrt{3}}y-\tfrac{1}{3}z\right)^2+\left(-\tfrac{\sqrt{2}}{3}x-\tfrac{\sqrt{2}}{\sqrt{3}}y-\tfrac{1}{3}z\right)^2\\ &= z^2+\tfrac{8}{9}x^2-\tfrac{4\sqrt{2}}{9}xz+\tfrac{1}{9}z^2+\left(\tfrac{\sqrt{2}}{3}x-\tfrac{\sqrt{2}}{\sqrt{3}}y+\tfrac{1}{3}z\right)^2+\left(\tfrac{\sqrt{2}}{3}x+\tfrac{\sqrt{2}}{\sqrt{3}}y+\tfrac{1}{3}z\right)^2\\ &= z^2+\tfrac{8}{9}x^2-\tfrac{4\sqrt{2}}{9}xz+\tfrac{1}{9}z^2+\left(\tfrac{\sqrt{2}}{3}x+\tfrac{1}{3}z\right)^2-2\left(\tfrac{\sqrt{2}}{3}x+\tfrac{1}{3}z\right)\left(\tfrac{\sqrt{2}}{\sqrt{3}}y\right)+\left(\tfrac{\sqrt{2}}{\sqrt{3}}y\right)^2\\ &\qquad +\left(\tfrac{\sqrt{2}}{3}x+\tfrac{1}{3}z\right)^2+2\left(\tfrac{\sqrt{2}}{3}x+\tfrac{1}{3}z\right)\left(\tfrac{\sqrt{2}}{\sqrt{3}}y\right)+\left(\tfrac{\sqrt{2}}{\sqrt{3}}y\right)^2\\ &= z^2+\tfrac{8}{9}x^2-\tfrac{4\sqrt{2}}{9}xz+\tfrac{1}{9}z^2+2\left(\tfrac{2}{9}x^2+\tfrac{2\sqrt{2}}{9}xz+\tfrac{1}{9}z^2\right)+2\times\tfrac{2}{3}y^2\\ &= z^2\left(1+\tfrac{1}{9}+\tfrac{2}{9}\right)+x^2\left(\tfrac{8}{9}+\tfrac{4}{9}\right)+y^2\left(\tfrac{4}{3}\right)\\ &= \tfrac{4}{3}\left(x^2+y^2+z^2\right)\\ &= \tfrac{4}{3} \qquad \text{since } x^2+y^2+z^z=1\end{aligned}$$

Hence:

$$\begin{aligned}\sum_{i=1}^{4}(XP_i)^4 &= 16+4\sum_{i=1}^{4}(\mathbf{p}_i.\mathbf{x})^2\\ &= 16+4\times\tfrac{4}{3}\end{aligned}$$

which is independent of $x, y$ and $z$.

# 2008 S3 Q4

## 13 Preparation

**(i)** The gradient is given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x}$, which means that the gradient is always positive and tends to 0 as $x \to \pm\infty$. We also have:

$$\tanh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}$$

As $x \to \infty$ we have $\tanh x \to 1$ and as $x \to -\infty$ we have $\tanh x \to -1$. This should be enough for you to sketch your graph and you can use Desmos to check your sketch.

**(ii)** First note that $\cosh^2\left(\frac{x}{2}\right) - \sinh^2\left(\frac{x}{2}\right) = 1$ and $\cosh^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{x}{2}\right) = \cosh x$. We then get:

$$\begin{aligned}
\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} &= \sqrt{\frac{\left(\cosh^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{x}{2}\right)\right) - \left(\cosh^2\left(\frac{x}{2}\right) - \sinh^2\left(\frac{x}{2}\right)\right)}{\left(\cosh^2\left(\frac{x}{2}\right) + \sinh^2\left(\frac{x}{2}\right)\right) + \left(\cosh^2\left(\frac{x}{2}\right) - \sinh^2\left(\frac{x}{2}\right)\right)}} \\
&= \sqrt{\frac{\cancel{2}\sinh^2\left(\frac{x}{2}\right)}{\cancel{2}\cosh^2\left(\frac{x}{2}\right)}} \\
&= \left|\tanh\left(\tfrac{x}{2}\right)\right| \\
&= \tanh\left(\tfrac{x}{2}\right) \qquad (x \geqslant 0)
\end{aligned}$$

If $x < 0$ we have $\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}} = -\tanh\left(\frac{x}{2}\right)$.

**(iii)** If $y = \text{arcosh}x$ then $\cosh y = x$. Differentiating this second equation with respect to x gives us:

$$\begin{aligned}
\sinh y \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sinh y} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{\cosh^2 y - 1}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{x^2 - 1}}
\end{aligned}$$

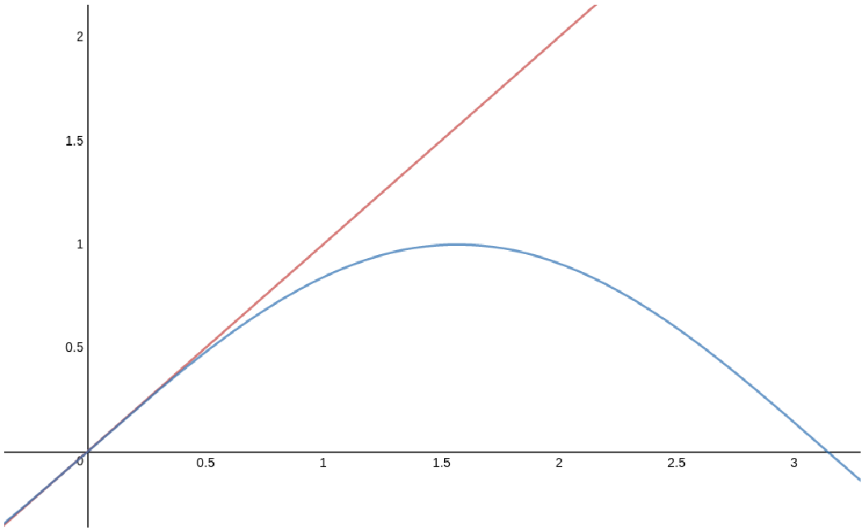
**(iv)**

$$\begin{aligned}\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x &= \int \frac{\sinh y}{\cancel{\sqrt{\sinh^2 y+1}}} \times \cancel{\cosh y}\,\mathrm{d}y \\ &= \int \sinh y\,\mathrm{d}y \\ &= \cosh y + k \\ &= \sqrt{x^2+1}+k\end{aligned}$$

To help us integrate directly first note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)^{\frac{1}{2}} = \frac{1}{2} \times 2x \times \left(x^2+1\right)^{-\frac{1}{2}}$. Hence:

$$\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x = \sqrt{x^2+1}+k\,.$$

**(v)** A quick sketch of $y = x$ and $y = \sin x$ (remember that $x$ is in radians) shows that $x > \sin(x)$ for $x > 0$.

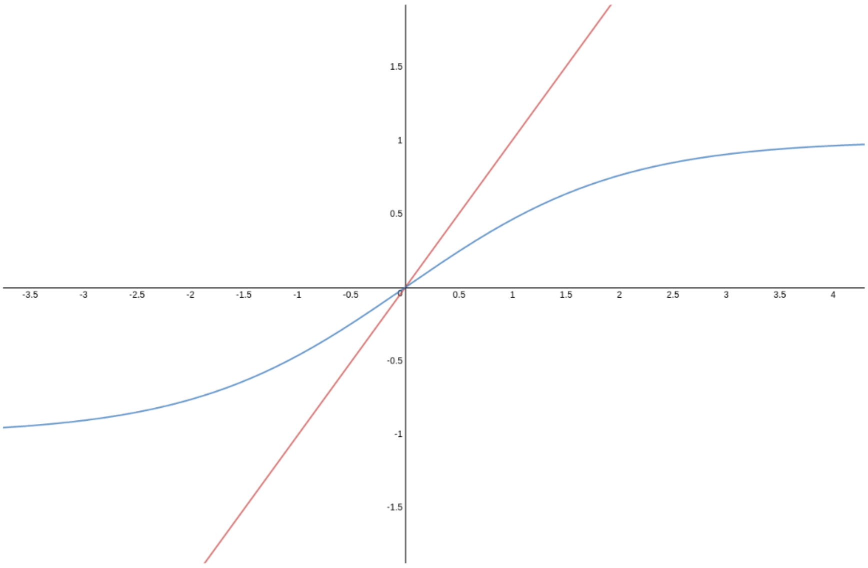



We then have:

$$\begin{aligned}\int_0^x t\,\mathrm{d}t &> \int_0^x \sin t\,\mathrm{d}t \\ \left[\tfrac{1}{2}t^2\right]_0^x &> \left[-\cos t\right]_0^x \\ \tfrac{1}{2}x^2 &> 1-\cos x\end{aligned}$$

## 14 The STEP III question

**(i)** Differentiating $\tanh\left(\frac{y}{2}\right)$ with respect to $y$ gives $\dfrac{1}{2\cosh^2\left(\frac{y}{2}\right)}$, so the gradient of $\tanh\left(\frac{y}{2}\right)$ is less than or equal to $\frac{1}{2}$. The graphs look like:




So when $y > 0$ we have $y > \tanh\left(\frac{y}{2}\right)$.

Letting $y = \text{arcosh}x$ (which needs $x > 1$) we have $x = \cosh y$ and:

$$\begin{aligned}\frac{x-1}{\sqrt{x^2-1}} &= \frac{\cosh y - 1}{\sqrt{\cosh^2 y - 1}}\\ &= \frac{\cosh y - 1}{\sinh y}\\ &= \frac{2\sinh^2\left(\frac{y}{2}\right) + 1 - 1}{2\sinh\left(\frac{y}{2}\right)\cosh\left(\frac{y}{2}\right)}\\ &= \frac{\sinh\left(\frac{y}{2}\right)}{\cosh\left(\frac{y}{2}\right)}\\ &= \tanh\left(\frac{y}{2}\right)\end{aligned}$$

Then since we have $y > \tanh\left(\frac{y}{2}\right)$ for $y > 0$ we have $\text{arcosh}x > \dfrac{x-1}{\sqrt{x^2-1}}$ for $x > 1$.

**(ii)** Integrating the LHS of $(*)$ gives:

$$\begin{aligned}
\int_1^x \text{arcosh}t \, \mathrm{d}t &= \int_0^{\text{arcosh}x} \theta \sinh\theta \, \mathrm{d}\theta && \text{using } t = \cosh\theta \\
&= \Big[\theta\cosh\theta\Big]_0^{\text{arcosh}x} - \int_0^{\text{arcosh}x} \cosh\theta \, \mathrm{d}\theta && \text{integration by parts} \\
&= x\text{arcosh}x - \Big[\sinh\theta\Big]_0^{\text{arcosh}x} \\
&= x\text{arcosh}x - \Big[\sqrt{\cosh^2\theta - 1}\Big]_0^{\text{arcosh}x} \\
&= x\text{arcosh}x - \sqrt{x^2-1}
\end{aligned}$$

Integrating the RHS of $(*)$ gives:

$$\begin{aligned}
\int_1^x \frac{t-1}{\sqrt{t^2-1}} \, \mathrm{d}t &= \int_0^{\text{arcosh}x} \frac{\cosh\theta - 1}{\sinh\theta} \times \sinh\theta \, \mathrm{d}\theta && \text{using } t = \cosh\theta \\
&= \Big[\sinh\theta - \theta\Big]_0^{\text{arcosh}x} \\
&= \sqrt{x^2-1} - \text{arcosh}x
\end{aligned}$$

Since $\mathrm{f}(x) > \mathrm{g}(x) \implies \int \mathrm{f}(x)\,\mathrm{d}x > \int \mathrm{g}(x)\,\mathrm{d}x$ we have:

$$\begin{aligned}
x\text{arcosh}x - \sqrt{x^2-1} &> \sqrt{x^2-1} - \text{arcosh}x \\
(x+1)\text{arcosh}x &> 2\sqrt{x^2-1} \\
\text{arcosh}x &> 2\frac{\sqrt{x^2-1}}{x+1} \\
\text{arcosh}x &> 2\frac{\sqrt{(x+1)(x-1)}}{x+1} \\
\text{arcosh}x &> 2\frac{\sqrt{x-1}}{\sqrt{x+1}} \\
\text{arcosh}x &> 2\frac{x-1}{\sqrt{(x+1)(x-1)}} \\
\text{arcosh}x &> 2\frac{x-1}{\sqrt{x^2-1}}
\end{aligned}$$

**(iii)** Now integrate the result from part **(ii)**. The LHS is as before and the RHS is twice what it was before. This gives us:

$$\begin{aligned}
x\text{arcosh}x - \sqrt{x^2-1} &> 2\left(\sqrt{x^2-1} - \text{arcosh}x\right) \\
(x+2)\text{arcosh}x &> 3\sqrt{x^2-1} \\
\text{arcosh}x &> 3\frac{\sqrt{x^2-1}}{x+2}
\end{aligned}$$

# 2013 S3 Q7

## 15 Preparation

**(i)** If $f'(x) \geqslant 0$ then f in increasing and so we have $f(x) \geqslant f(0)$ (and for this function this means $f(x) \geqslant 0$) for $x \geqslant 0$.

**(ii)** **(a)** $\cos x e^{\sin x}$

**(b)** $2x \sinh\left(x^2\right)$

**(c)** $f'(x)\cosh\left(f(x)\right)$

**(d)** $xf''(x) + f'(x)$

**(e)** $2f'(x)f''(x)$

**(iii)** Integrating (by parts for the first term) gives:

$$\begin{aligned}\int (Ax\sin x + B\cos x)\ dx &= [-Ax\cos x] + \int A\cos x\, dx + B\sin x \\ &= A(\sin x - x\cos x) + B\sin x + k\end{aligned}$$

**(iv)** If we let $f(x) = y$ then $\dfrac{dy}{dx} = f'(x)$. We have:

$$\begin{aligned}\int f(x)e^{f(x)}f'(x)\, dx &= \int ye^y\, dy \\ &= ye^y - e^y + k \\ &= f(x)e^{f(x)} - e^{f(x)} + k\end{aligned}$$

**(v)** Considering $\cosh x - 1$ we have:

$$\begin{aligned}\cosh x - 1 &= \frac{e^x + e^{-x} - 2}{2} \\ &= \frac{\left(e^{x/2} - e^{-x/2}\right)^2}{2} \\ &\geqslant 0\end{aligned}$$

and hence we have $\cosh x \geqslant 1$

When asked to show an inequality it is often easier to rearrange and show greater than or less than 0. There are other ways you can approach this question, including differentiation to find the minimum point.

## 16 The STEP III question

**(i)** Differentiating $\mathrm{E}(x)$ gives:

$$\begin{aligned}\mathrm{E}'(x) &= 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + 2y^3\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ &= 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right) + y^3\right)\\ &= 0\end{aligned}$$

Hence $\mathrm{E}(x)$ is constant.

Since $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ we have $\mathrm{E}(0) = \frac{1}{2}$ and so

$$y^4 = 2\mathrm{E}(x) - 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \leqslant 1\,.$$

**(ii)** Differentiating $\mathrm{E}(x)$ gives:

$$\begin{aligned}\frac{\mathrm{dE}}{\mathrm{d}x} &= 2\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\right) + 2\sinh v\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\\ &= 2\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2v}{\mathrm{d}x^2} + \sinh v\right)\\ &= 2\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\left(-x\frac{\mathrm{d}v}{\mathrm{d}x}\right)\\ &= -2x\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2\\ \therefore \frac{\mathrm{dE}}{\mathrm{d}x} &\leqslant 0 \qquad \text{for } x \geqslant 0\end{aligned}$$

We now know that $\mathrm{E}(x)$ is decreasing for $x \geqslant 0$, so we have

$$\mathrm{E}(x) \leqslant \mathrm{E}(0) = 0^2 + 2\cosh(\ln 3)$$

Since $2\cosh(\ln 3) = \mathrm{e}^{\ln 3} + \mathrm{e}^{-\ln 3} = \frac{10}{3}$ we have:

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v \leqslant \frac{10}{3}$$

and hence $\cosh v \leqslant \frac{5}{3}$ as required.

**(iii)** Here you are left on your own. Comparing to the previous parts suggests that it would be a good idea to consider an "$\mathrm{E}(x)$".

Let $\mathrm{E}(x) = \left(\dfrac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2(w\sinh w + \cosh w)$. We can find this by comparing to the other two functions — the first term is the square of the relevant derivative and the last term is twice the integral of the last term in the given differential equation.

Differentiating $\mathrm{E}(x)$ gives us:

$$\begin{aligned}\frac{\mathrm{dE}}{\mathrm{d}x} &= 2\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) + 2\left(2\sinh w + w\cosh w\right)\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\\ &= 2\frac{\mathrm{d}w}{\mathrm{d}x}\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + 2\sinh w + w\cosh w\right)\\ &= 2\frac{\mathrm{d}w}{\mathrm{d}x}\left(-(5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x}\right)\\ &= -\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \times 2(5\cosh x - 4\sinh x - 3)\end{aligned}$$

Considering the last set of brackets here we have:

$$\begin{aligned}2(5\cosh x - 4\sinh x - 3) &= 5(\mathrm{e}^x + \mathrm{e}^{-x}) - 4(\mathrm{e}^x - \mathrm{e}^{-1}) - 6\\ &= \mathrm{e}^x + 9\mathrm{e}^{-x} - 6\\ &= \mathrm{e}^{-x}\left(\mathrm{e}^{2x} - 6\mathrm{e}^x + 9\right)\\ &= \mathrm{e}^{-x}\left(\mathrm{e}^x - 3\right)^2 \geqslant 0\end{aligned}$$

Hence we have $\mathrm{E}'(x) \leqslant 0$. We also have $\mathrm{E}(0) = \frac{1}{2} + 2 = \frac{5}{2}$, and so $\mathrm{E}(x) \leqslant \frac{5}{2}$ for $x \geqslant 0$. This gives:

$$\begin{aligned}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2(w\sinh w + \cosh w) &\leqslant \frac{5}{2}\\ w\sinh w + \cosh w &\leqslant \frac{5}{4}\end{aligned}$$

and as $w\sinh w \geqslant 0$ (since if $w < 0$, $\sinh w < 0$ etc) we have $\cosh w \leqslant \frac{5}{4}$ as required.