STEP Support Programme

STEP 2 Curve Sketching Questions: Solutions

These are not fully worked solutions — you need to fill in some gaps. They seem rather long in places as there is quite a lot of discussion along the way. For clarity, these graphs have been drawn by computer but you would be expected to *sketch* the graphs, not *plot* them (which is generally frowned upon in STEP — roughly plotting a couple of important points is OK, but not working out lots of values of y for different values of x and then joining them up).

1 To differentiate you will need the product rule (and the chain rule for $(x-2)^4$). Using this gives:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x+1)(x-2)^4 + x(x-2)^4 + 4x(x+1)(x-2)^3 \\ &= (x-2)^3(2x-1)(3x+2)\,.\end{aligned}$$

So the x coordinates are $x=2$, $x=\frac{1}{2}$ and $x=-\frac{2}{3}$.

The y coordinates are not nice for two of these — but if you want you could leave them in index form, so the turning points are at $(2,0)$, $\left(\frac{1}{2}, \frac{3^5}{2^6}\right)$ and $\left(-\frac{2}{3}, -\frac{2^{13}}{3^6}\right)$.

To classify these you can look at the second derivative,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3(x-2)^2(2x-1)(3x+2) + 2(x-2)^3(3x+2) + 3(x-2)^3(2x-1)\,.$$

You can simplify this further, but it is actually not very helpful to do so!

When $x=2$, you can see that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$. This does **not** mean that the point $(2,0)$ is a point of inflection! We need to think more carefully about this one, so we will park it for now.

When $x=\frac{1}{2}$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 + 2\times\left(\frac{1}{2}-2\right)^3 \times \left(3\times\frac{1}{2}+2\right) + 0$$

and since this is "something positive" × "something negative" × "something positive" (plus some zeros), the second derivative is negative and hence this point is a maximum.

When $x=-\frac{2}{3}$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 + 0 + 3\times\left(-\frac{2}{3}-2\right)^3 \times \left(2\times -\frac{2}{3}-1\right)$$

and since this is "something positive" × "something negative" × "something negative" (plus some zeros), the second derivative is positive and hence this point is a minimum.

By setting $y = 0$ you can show that the curve crosses the x-axis at $(0, 0)$, $(-1, 0)$ and $(2, 0)$. By noting that for "large" x the graph is similar to $y = x^6$ you can deduce that as $x \to \pm\infty$, $y \to +\infty$. These bits of information, along with the minimum at $x = -\frac{2}{3}$ and maximum at $x = \frac{1}{2}$ are enough to sketch the curve and work out what sort of point $(2, 0)$ is (remember there can be no more turning points!).

To formally work out what sort of point $(2, 0)$ is we can look at the sign of the gradient either side of this point. You can substitute $x = 1.9$ and $x = 2.1$ if you like, but this might be a messy without a calculator.

A different approach is to consider $x = 2 - \epsilon$ and $x = 2 + \epsilon$, where ϵ is "small" and positive (we can write $\epsilon > 0$, $\epsilon \ll 1$ where the second statement means "ϵ is much smaller than 1").

Now considering $x = 2 - \epsilon$ we have:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x-2)^3(2x-1)(3x+2) \\ &= (-\epsilon)^3\,(3-2\epsilon)\,(8-3\epsilon)\end{aligned}$$

which is the product of one negative bracket and two positive ones (remember that ϵ is small) and so to the left of the point $(2, 0)$ the gradient is negative.

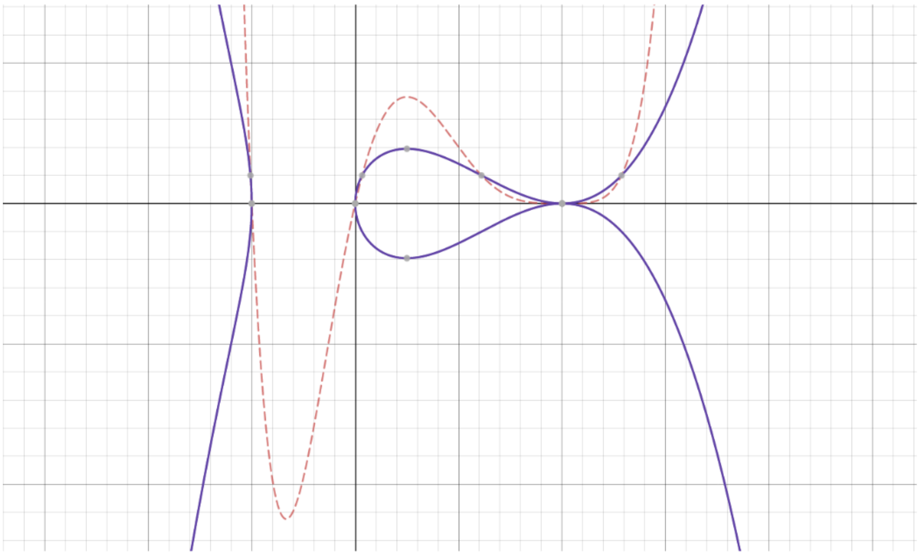
For $x = 2 + \epsilon$ we have:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= (x-2)^3(2x-1)(3x+2) \\ &= (\epsilon)^3\,(3+2\epsilon)\,(8+3\epsilon)\end{aligned}$$

which is the product of three positive brackets and so to the right of the point $(2, 0)$ the gradient is positive.

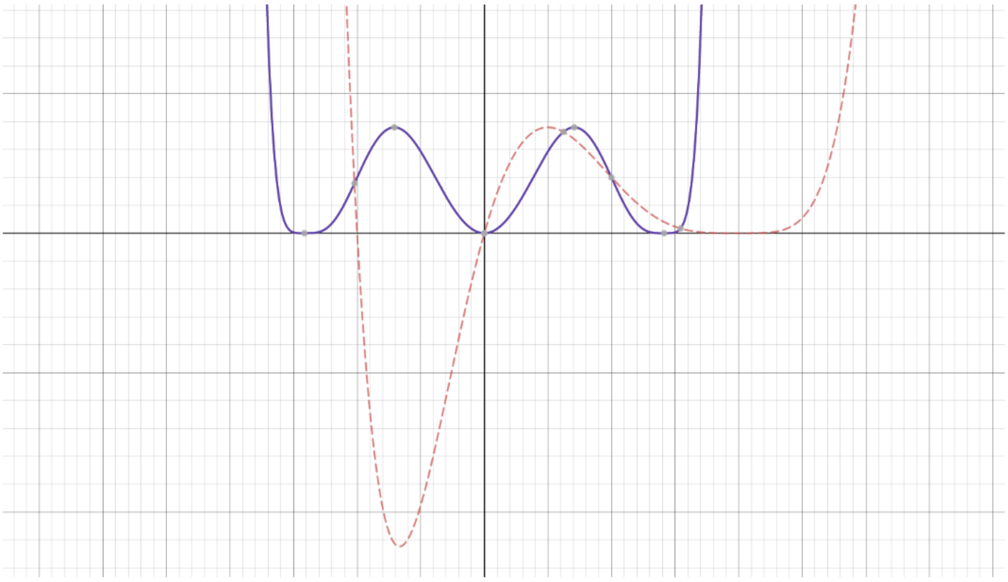
This means that near the point $(2, 0)$ the graph looks like $\searrow\!\!\text{--}\!\!\nearrow$ and so the point $(2, 0)$ is a minimum.

(i) The graph below shows both $y^2 = x(x+1)(x-2)^4$ and $y = x(x+1)(x-2)^4$, the second one being dotted.



You can see that the curves intersect whenever the y coordinate is 1 or 0, that the curve $y^2 = \mathrm{f}(x)$ is symmetrical about the x-axis and that the curve $y^2 = \mathrm{f}(x)$ does not exist when $y = \mathrm{f}(x)$ is negative.

(ii) The graph below shows both $y = x^2(x^2+1)(x^2-2)^4$ and $y = x(x+1)(x-2)^4$, the second one being dotted.



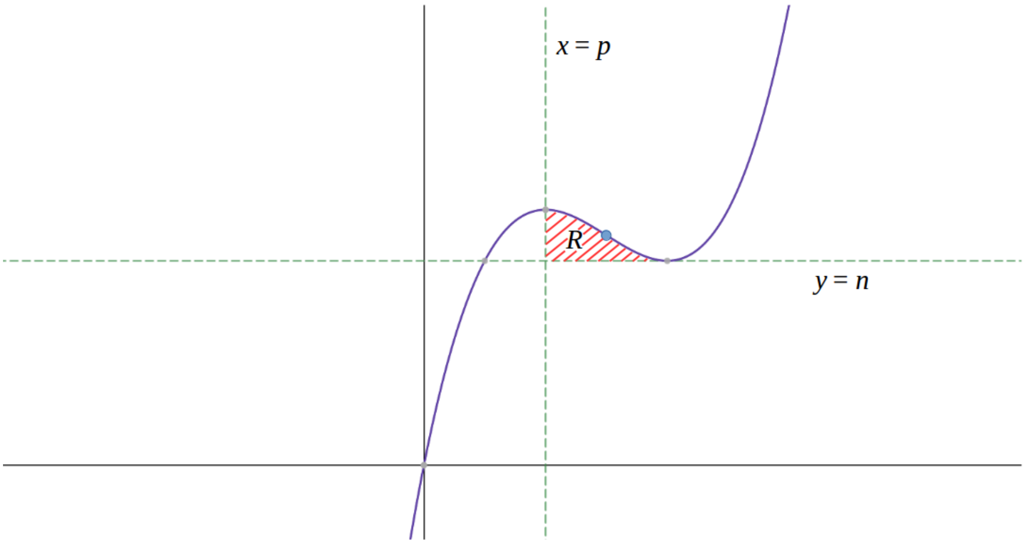
The stationary points at $x = \frac{1}{2}$ and $x = 2$ have moved so that they are now at $x = \sqrt{\frac{1}{2}}$ and $x = \sqrt{2}$, and the right hand side of the new graph looks like a squashed/stretched version of the original graph. The new graph has reflectional symmetry about the y-axis.

2 This question caused some problems as the region "R" is not uniquely defined - there are two possible regions! When it was set, all those who picked the "wrong" region had their work marked by the Principal Examiner to make sure that they were not disadvantaged.

(i) Differentiation gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 2bx + c \equiv k(x-p)(x-q)$. By equating coefficients we get $k = 6$, $b = 3(p+q)$ and $c = 6pq$.

(ii) This is a cubic, which passes through $(0,0)$. The conditions $p > 0$ and $n > 0$ mean that both of the turning points lie in the first quadrant. The point of inflection lies half way between the turning points[1]. Note that this is a *non-stationary* point of inflection, i.e. the gradient of the curve is not zero here.

The sketch should look something like:



The curve has rotational symmetry order 2 about the point of inflection.

(iii) $m - n = (2p^3 - bp^2 + cp) - (2q^3 - bq^2 + cq) = 2(p^3 - q^3) - b(p^2 - q^2) + c(p-q)$. The best starting point is probably to factorise out $(p-q)$ — or even $(q-p)$ — and then substitute in your answers for b and c.

The difference of two cubes identity, $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$, will be helpful (and is well worth knowing).

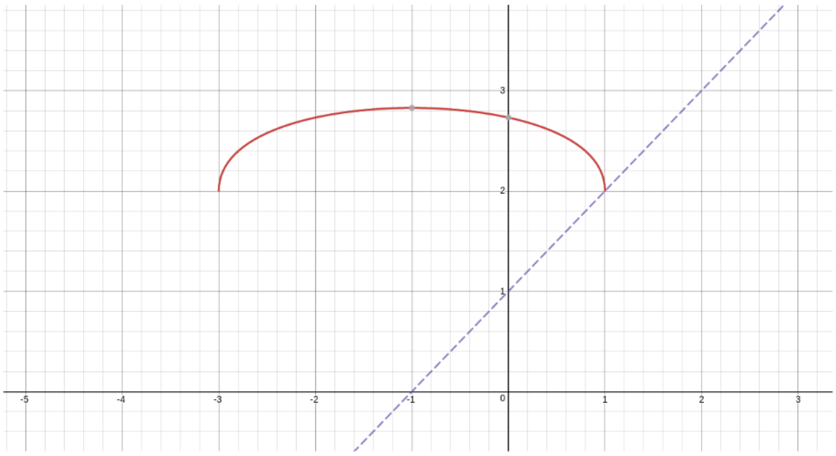
(iv) The last bit uses the fact that the shaded is region is half of a rectangle which is $p-q$ wide and $m-n$ high, along with the result from part **(iii)**.

[1] It is not necessary to do so, but if you set the second derivative equal to zero you will find that the x coordinate of the point of point of inflection is $x = \frac{b}{6} = \frac{p+q}{2}$, i.e. half way between $x = p$ and $x = q$.

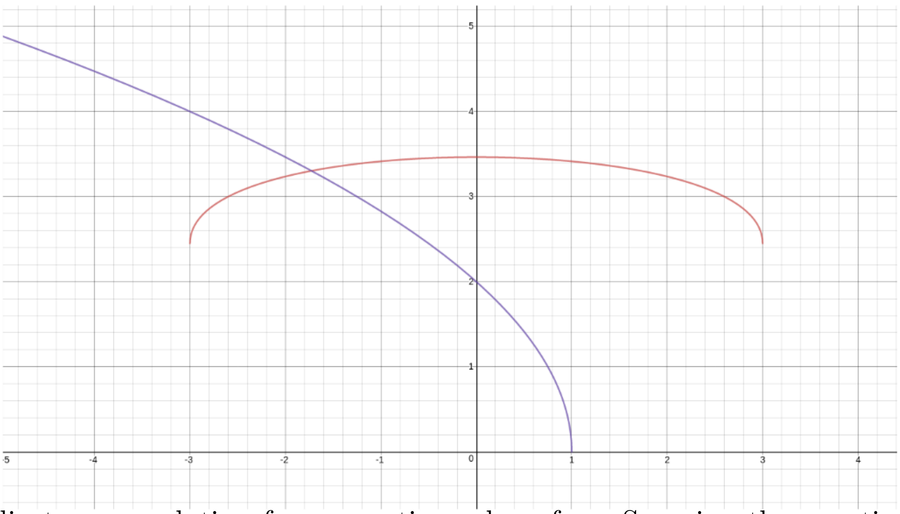
3 **(i)** This curve is only defined for $-3 \leqslant x \leqslant 1$, so it starts at $(-3, 2)$ and ends at $(1, 2)$. It cannot intersect the x axis as then we would need $\sqrt{1-x} + \sqrt{3+x} = 0$ which can only be true when both square roots are zero. It intersects the y axis at $(0, 1 + \sqrt{3})$.

The derivative of the graph is given by $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{3+x}} - \dfrac{1}{2\sqrt{1-x}}$ which is zero for $x = -1$. The y coordinate here is $2\sqrt{2}$, and so this must be a maximum as the y value here is greater than the y value at the end points. The derivative also tells us that the gradient of the graph approaches infinity as you approach the end points.

Sketching on $y = x + 1$ will show that there is only one solution at $x = 1$, where the straight line meets one of the endpoints of the curve.



(ii) Sketching the graphs will result in a picture like the following (you should find the coordinates of the end points and the maximum):



This indicates one solution for a negative value of x. Squaring the equation gives $4(1-x) = 3 + x + 3 - x + 2\sqrt{9-x^2}$. Rearranging to get $\sqrt{9-x^2} = -1 - 2x$ and squaring gives $5x^2 + 4x - 8 = 0$ which can be solved to give $x = \frac{-2-2\sqrt{11}}{5}$ (we know that x is negative, so we take the negative root only)[2].

[2]Squaring an equation usually introduces "extra" solutions so it is always a good idea to check that your solutions are possible. Usually you can do this by substituting into the original equation, or by using a sketch to justify the solution(s).

4 Differentiation gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{x^2-2x+a} - x \times \frac{1}{2}(2x-2)(x^2-2x+a)^{-\frac{1}{2}}}{x^2-2x+a}\,.$$

Which can be simplified to give:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a-x}{(x^2-2x+a)^{\frac{3}{2}}}\,,$$

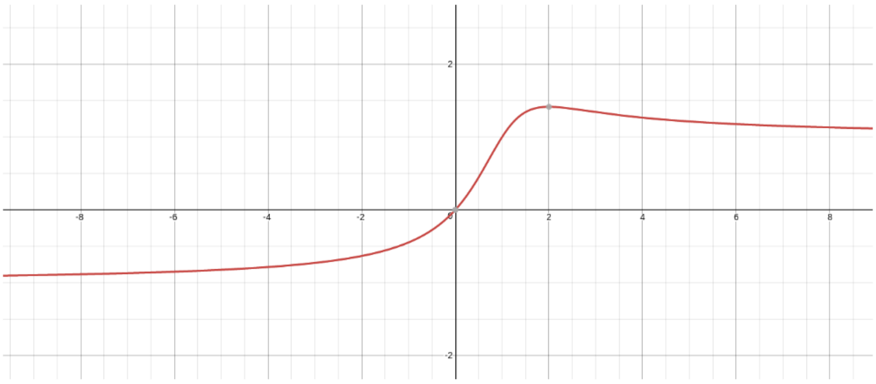
and this can only be zero when $x = a$.

The condition $a > 1$ means that x^2-2x+a cannot be zero (as the discriminant of $x^2-2x+a = 0$ is $4-4a < 0$ for $a > 1$). So when $a > 1$, $x = a$ is a stationary point.

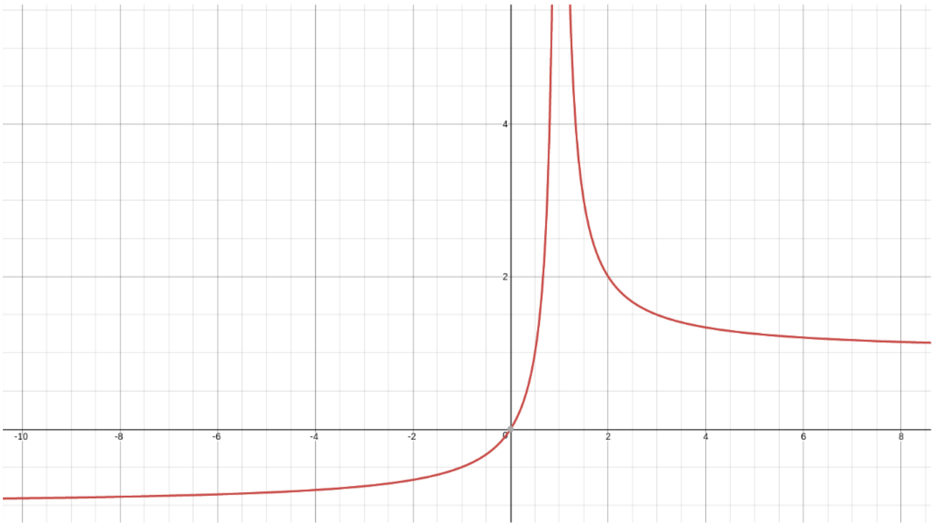
Note that for some values of $a \leqslant 1$ the graph has a stationary point whereas for others it does not, depending on whether a lies in the range of x for which $x^2-2x+a \leqslant 0$ or not.

(i) The only place where this graph crosses the axes is $(0, 0)$. As $x \to \pm\infty$, we can say $\sqrt{x^2-2x+2} \approx \sqrt{x^2} = |x|$, so as $x \to +\infty, y \to 1$ and as $x \to -\infty, y \to -1$. There are no vertical asymptotes as x^2-2x+2 cannot be zero (the discriminant is equal to $4-8 = -4$ in this case).

From before, there is exactly one stationary point at $x = 2$ (when $y = \sqrt{2}$). Putting this all together gives a graph something like:



(ii) When $a = 1$ we have $y = \frac{x}{\sqrt{(x-1)^2}} = \frac{x}{|x-1|}$. This passes through the origin and approaches $y = \pm 1$ as $x \to \pm\infty$ as in part **(ii)**. However, as $x \to 1$, $y \to \infty$, i.e. we have a vertical asymptote at $x = 1$.



Using Desmos you can plot $y = \dfrac{x}{\sqrt{x^2 - 2x + a}}$ and use a "slider" to see what happens as a varies. You may like to include the line $x = a$ as well.