STEP Support Programme

STEP 2 Curve Sketching Topic Notes

When sketching a curve, consider the following:

- Where the curve intercepts the x and y axes. To do this, substitute $y = 0$ and $x = 0$ respectively and solve for x and y.
- The location and nature of the turning points.
- The location of any *points of inflection*. These are where the "bend" of the curve changes, and satisfy $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ **as well as** $\frac{\mathrm{d}y}{\mathrm{d}x}$ having the same sign either side of the point of inflection. See here for another definition of a point of inflection.
- The location of any *asymptotes*. When $x \to \pm\infty$ we can get horizontal or oblique (slanted) asymptotes. When x approaches some values we may get $y \to \pm\infty$ — there are vertical asymptotes here.
- What the behaviour is like as x gets close to zero.

There are other things you can consider — for example is the curve you are trying to sketch a transformation (especially a translation) of an easier curve?

Can you deduce anything from the gradient? For example if the gradient is $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{1}{x^2}$ then you know the gradient is positive and the graph is always "going upwards".

Is the graph even (i.e. is $\mathrm{f}(-x) = \mathrm{f}(x)$)? If this is the case then we must have reflection symmetry in the y-axis. Or is the graph odd (when $\mathrm{f}(-x) = -\mathrm{f}(x)$)? In this case we have rotational symmetry order 2 about the origin. **Note: functions may be neither odd nor even**.

$y = \mathrm{f}(x^2)$

If you have sketched $y = \mathrm{f}(x)$ you can use this to sketch $y = \mathrm{f}(x^2)$. If there are any points on $y = \mathrm{f}(x)$ where the curve intersects the x-axis or where there is a turning point, and the x-coordinate is $a \geqslant 0$ here, then there will be an intersection or turning point on $y = \mathrm{f}(x^2)$ with x-coordinate equal to $\sqrt{a}$. Furthermore, all the points (a, b) $(a \geqslant 0)$ will transform to the point $(\sqrt{a}, b)$, so the graph looks more and more "squashed" as x tends to infinity. Finally the graph of $y = \mathrm{f}(x^2)$ will be even, so when you have drawn it for positive x you can reflect in the y-axis.

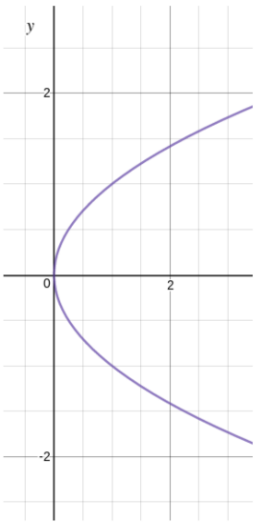
$y^2 = \mathrm{f}(x)$

Since $y^2 \geqslant 0$ you need to restrict yourself to values of x for which $\mathrm{f}(x) \geqslant 0$. Then consider the effect of square-rooting $\mathrm{f}(x)$. If $0 < \mathrm{f}(x) < 1$ then $\sqrt{\mathrm{f}(x)} > \mathrm{f}(x)$ but if $\mathrm{f}(x) > 1$ then $\sqrt{\mathrm{f}(x)} < \mathrm{f}(x)$. When $\mathrm{f}(x) = 0$ or 1 the curves $y = \mathrm{f}(x)$ and $y^2 = \mathrm{f}(x)$ meet. Finally, if $y^2 = \mathrm{f}(x)$ then $y = \pm\sqrt{\mathrm{f}(x)}$, so reflect your curve segments in the x-axis.

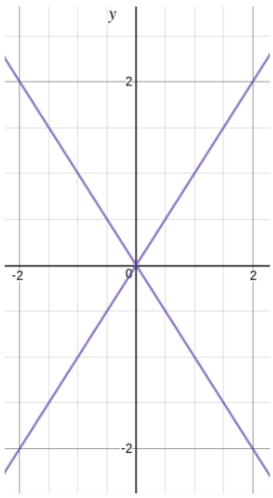
You might like to try sketching $y = \sin x$, $y = \sin(x^2)$ and $y^2 = \sin x$. Use Desmos to check your graphs.

Top Tips!

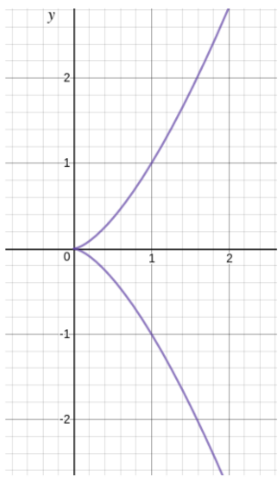
- Draw it large!
- Check the domain of the function; where is it defined? Especially important for sketches of $y^2 = f(x)$; need $f(x) \geq 0$.
- Look for asymptotes; vertical, horizontal and oblique, and in general look for the behaviour for large x. Does the function grow or decay or approach a constant?
- For a rational function, use polynomial division to reduce the order of the numerator, e.g.
$$y = \frac{x^2+x-1}{x-1} = \frac{x(x-1)+2x-1}{x-1} = x + \frac{2(x-1)+1}{x-1} = x+2+\frac{1}{x-1}$$
so for large (and positive) x, $y \approx x+2$, and the remainder term is positive so the curve lies above this oblique asymptote. For large negative x the remainder term is negative, so the curve lies below this asymptote. By setting $\frac{x^2+x-1}{x-1} = x+2$ and trying to solve for x you can show that the curve does not meet the asymptote.
- Are there any other interesting points?
- Is there any symmetry? For example, if (x, y) is a point on the curve $y^2 = f(x)$, then $(x, -y)$ is also a point on the curve, giving reflective symmetry in the x-axis.
- What's the sign of the function of each side of its vertical asymptote? Is is positive on one side, negative on the other (like $y = 1/x$) or positive on both sides (like $y = 1/x^2$)?
- What does the curve look like when it reaches the x-axis? Important for curves $y^2 = f(x)$, where the gradient is $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)}{2y}$. We're at a point with $y = 0$, so typically the gradient will be infinite. But if $f'(x)$ is also zero then the gradient might not go to infinity. To see what happens in some special cases, sketch $y^2 = x$, $y^2 = x^2$ and $y^2 = x^3$. In general, you'll need to work out which of these functions your curve looks like near the x-axis — using $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f'(x)}{2y}$ to write the gradient in terms of x can be useful.



$y^2 = x$
Infinite gradient
at (0,0)
$\frac{dy}{dy} = \frac{1}{2y}$



$y^2 = x^2$
Gradient = 1
at (0,0)
$\frac{dy}{dy} = \frac{2x}{2y} = \pm 1$



$y^2 = x^3$
Gradient = 0
at (0,0)
$\frac{dy}{dy} = \frac{3x^2}{2y} = \pm\frac{3}{2}\sqrt{x}$

More notes

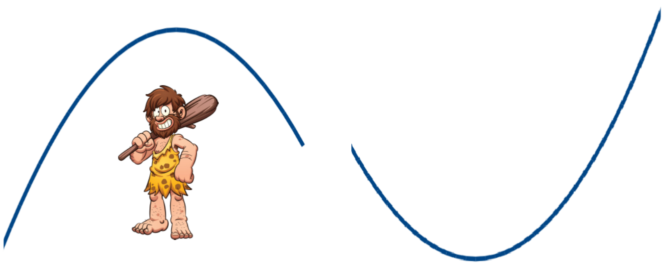
Turning points

To find these, use $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. You may have to use any or all of the product rule, quotient rule, chain rule, implicit differentiation etc. **Do not** divide throughout by something that might be zero (such as x) as then you will lose a turning point. However, you can divide by e^x as e^x cannot be zero.

You can use the second derivative to tell if a turning point is a maximum or minimum $\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0 \implies \text{minimum}\right)$ but $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ does not mean that the point is a point of inflection. To classify these points you can look at the sign of the gradient on each side of the point.

Points of Inflection

A point of inflection is one where the "bend" of the curve changes, i.e. the curve changes from being concave to convex or vice-versa. A con**cave** graph has $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, and a **cave**man can live under it.



Concave
Gradient is decreasing

Convex
Gradient is increasing

At a point of inflection we have $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ as well as the **gradient having the same sign** either side of this point. If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ as well then we have a *stationary* point of inflection, otherwise it is a *non-stationary* point of inflection. In the sketch below the non-stationary point of inflection is where the graph is changing from convex to concave and the stationary point of inflection is where the graph is changing from concave to convex.