STEP Support Programme

STEP 2 Probability and Statistics Questions

1 **2003 S2 Q13**

The random variable X takes the values $k = 1, 2, 3, \ldots,$ and has probability distribution

$$\mathrm{P}(X = k) = A\frac{\lambda^k \mathrm{e}^{-\lambda}}{k!},$$

where λ is a positive constant. Show that $A = (1 - \mathrm{e}^{-\lambda})^{-1}$. Find the mean μ in terms of λ and show that

$$\mathrm{Var}\,(X) = \mu(1 - \mu + \lambda)\ .$$

Deduce that $\lambda < \mu < 1 + \lambda$.

Use a normal approximation to find the value of $P(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

2 **2007 S2 Q14**

The random variable X has a continuous probability density function $\mathrm{f}(x)$ given by

$$\mathrm{f}(x) = \begin{cases} 0 & \text{for } x \leqslant 1 \\ \ln x & \text{for } 1 \leqslant x \leqslant k \\ \ln k & \text{for } k \leqslant x \leqslant 2k \\ a - bx & \text{for } 2k \leqslant x \leqslant 4k \\ 0 & \text{for } x \geqslant 4k \end{cases}$$

where k, a and b are constants.

(i) Sketch the graph of $y = \mathrm{f}(x)$.

(ii) Determine a and b in terms of k and find the numerical values of k, a and b.

(iii) Find the median value of X.

3 **2011 S2 Q12**

Xavier and Younis are playing a match. The match consists of a series of games and each game consists of three points.

Xavier has probability p and Younis has probability $1-p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1-p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let w be the probability that Younis wins the match. Show that, for $p \neq 0$,

$$w = \frac{1-p^2}{2-p}.$$

Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does w increase whenever p decreases?

(ii) If Xavier wins the match, Younis gives him £1; if Younis wins the match, Xavier gives him £k. Find the value of k for which the game is 'fair' in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

4 **2012 S2 Q13**

In this question, you may assume that $\displaystyle\int_0^\infty e^{-x^2/2}\,dx = \sqrt{\tfrac{1}{2}\pi}\,.$

The number of supermarkets situated in any given region can be modelled by a Poisson random variable, where the mean is k times the area of the given region. Find the probability that there are no supermarkets within a circle of radius y.

The random variable Y denotes the distance between a randomly chosen point in the region and the nearest supermarket. Write down $P(Y < y)$ and hence show that the probability density function of Y is $2\pi yk e^{-\pi ky^2}$ for $y \geqslant 0$.

Find $E(Y)$ and show that $\text{Var}\,(Y) = \dfrac{4-\pi}{4\pi k}$.

5 **2010 S1 Q13**

The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$pe^{2\lambda} - e^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

Discussion

Note that $\lambda > 0$ if and only if $e^{\lambda} > 1$. This is an old STEP I question from when the Poisson distribution was on the STEP I specification.

6 **2010 S2 Q13**

The continuous random variable X has probability density function $f(x)$, where

$$f(x) = \begin{cases} a & \text{for } 0 \leqslant x < k \\ b & \text{for } k \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$E(X) = \frac{1 - 2b + ab}{2(a - b)}.$$

(ii) Show that the median, M, of X is given by $M = \frac{1}{2a}$ if $a + b \geqslant 2ab$ and obtain an expression for the median if $a + b \leqslant 2ab$.

(iii) Show that $M < E(X)$.