

STEP Support Programme

STEP 2 Probability and Statistics Questions

1 2003 S2 Q13

The random variable X takes the values $k = 1, 2, 3, \dots$, and has probability distribution

$$P(X = k) = A \frac{\lambda^k e^{-\lambda}}{k!},$$

where λ is a positive constant. Show that $A = (1 - e^{-\lambda})^{-1}$. Find the mean μ in terms of λ and show that

$$\text{Var}(X) = \mu(1 - \mu + \lambda).$$

Deduce that $\lambda < \mu < 1 + \lambda$.

Use a normal approximation to find the value of $P(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

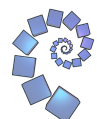
2 2007 S2 Q14

The random variable X has a continuous probability density function $f(x)$ given by

$$f(x) = \begin{cases} 0 & \text{for } x \leq 1 \\ \ln x & \text{for } 1 \leq x \leq k \\ \ln k & \text{for } k \leq x \leq 2k \\ a - bx & \text{for } 2k \leq x \leq 4k \\ 0 & \text{for } x \geq 4k \end{cases}$$

where k , a and b are constants.

- (i) Sketch the graph of $y = f(x)$.
- (ii) Determine a and b in terms of k and find the numerical values of k , a and b .
- (iii) Find the median value of X .



3 2011 S2 Q12

Xavier and Younis are playing a match. The match consists of a series of games and each game consists of three points.

Xavier has probability p and Younis has probability $1 - p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1 - p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let w be the probability that Younis wins the match. Show that, for $p \neq 0$,

$$w = \frac{1 - p^2}{2 - p}.$$

Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does w increase whenever p decreases?

(ii) If Xavier wins the match, Younis gives him $\mathcal{L}1$; if Younis wins the match, Xavier gives him $\mathcal{L}k$. Find the value of k for which the game is 'fair' in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

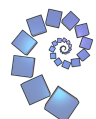
4 2012 S2 Q13

In this question, you may assume that $\int_0^\infty e^{-x^2/2} dx = \sqrt{\frac{1}{2}\pi}$.

The number of supermarkets situated in any given region can be modelled by a Poisson random variable, where the mean is k times the area of the given region. Find the probability that there are no supermarkets within a circle of radius y .

The random variable Y denotes the distance between a randomly chosen point in the region and the nearest supermarket. Write down $P(Y < y)$ and hence show that the probability density function of Y is $2\pi y k e^{-\pi k y^2}$ for $y \geq 0$.

Find $E(Y)$ and show that $\text{Var}(Y) = \frac{4 - \pi}{4\pi k}$.



5 2010 S1 Q13

The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p , show that

$$pe^{2\lambda} - e^\lambda + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p , find an expression for $\lambda_1 + \lambda_2$ in terms of p .

Find the probability, in terms of p , that she waits between 1 and 2 hours in the morning to receive her first text.

Discussion

Note that $\lambda > 0$ if and only if $e^\lambda > 1$. This is an old STEP I question from when the Poisson distribution was on the STEP I specification.

6 2010 S2 Q13

The continuous random variable X has probability density function $f(x)$, where

$$f(x) = \begin{cases} a & \text{for } 0 \leq x < k \\ b & \text{for } k \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$E(X) = \frac{1 - 2b + ab}{2(a - b)}.$$

(ii) Show that the median, M , of X is given by $M = \frac{1}{2a}$ if $a + b \geq 2ab$ and obtain an expression for the median if $a + b \leq 2ab$.

(iii) Show that $M < E(X)$.

