1 2003 S2 Q13
The random variable $X$ takes the values $k = 1, 2, 3, \ldots$, and has probability distribution
\[ P(X = k) = A \frac{\lambda^k e^{-\lambda}}{k!}, \]
where $\lambda$ is a positive constant. Show that $A = (1 - e^{-\lambda})^{-1}$. Find the mean $\mu$ in terms of $\lambda$ and show that
\[ \text{Var}(X) = \mu(1 - \mu + \lambda). \]
Deduce that $\lambda < \mu < 1 + \lambda$.
Use a normal approximation to find the value of $P(X = \lambda)$ in the case where $\lambda = 100$, giving your answer to 2 decimal places.

2 2007 S2 Q14
The random variable $X$ has a continuous probability density function $f(x)$ given by
\[ f(x) = \begin{cases} 
0 & \text{for } x \leq 1 \\
\ln x & \text{for } 1 \leq x \leq k \\
\ln k & \text{for } k \leq x \leq 2k \\
a - bx & \text{for } 2k \leq x \leq 4k \\
0 & \text{for } x \geq 4k
\end{cases} \]
where $k, a$ and $b$ are constants.

(i) Sketch the graph of $y = f(x)$.

(ii) Determine $a$ and $b$ in terms of $k$ and find the numerical values of $k, a$ and $b$.

(iii) Find the median value of $X$. 

STEP 2 Statistics Questions
3 2011 S2 Q12
Xavier and Younis are playing a match. The match consists of a series of games and each
game consists of three points.
Xavier has probability $p$ and Younis has probability $1 - p$ of winning the first point of any
game. In the second and third points of each game, the player who won the previous point
has probability $p$ and the player who lost the previous point has probability $1 - p$ of winning
the point. If a player wins two consecutive points in a single game, the match ends and that
player has won; otherwise the match continues with another game.

(i) Let $w$ be the probability that Younis wins the match. Show that, for $p \neq 0$,
$$w = \frac{1 - p^2}{2 - p}.$$  
Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does $w$ increase whenever $p$ decreases?

(ii) If Xavier wins the match, Younis gives him £1; if Younis wins the match, Xavier gives
him £$k$. Find the value of $k$ for which the game is ‘fair’ in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

4 2012 S2 Q13
In this question, you may assume that \( \int_0^\infty e^{-x^2/2}dx = \sqrt{\frac{1}{2\pi}}. \)
The number of supermarkets situated in any given region can be modelled by a Poisson
random variable, where the mean is $k$ times the area of the given region. Find the probability
that there are no supermarkets within a circle of radius $y$.
The random variable $Y$ denotes the distance between a randomly chosen point in the region
and the nearest supermarket. Write down $P(Y < y)$ and hence show that the probability
density function of $Y$ is $2\pi yk e^{-\pi ky^2}$ for $y \geq 0$.
Find $E(Y)$ and show that $\text{Var}(Y) = \frac{4 - \pi}{4\pi k}$.
5 2010 S1 Q13
The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean $\lambda$ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is $p$, show that

$$pe^{2\lambda} - e^\lambda + 1 = 0.$$ 

Given that $4p < 1$, show that there are two positive values of $\lambda$ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means $\lambda_1$ and $\lambda_2$ texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also $p$, find an expression for $\lambda_1 + \lambda_2$ in terms of $p$.

Find the probability, in terms of $p$, that she waits between 1 and 2 hours in the morning to receive her first text.

Discussion
Note that $\lambda > 0$ if and only if $e^\lambda > 1$. This is an old STEP I question from when the Poisson distribution was on the STEP I specification.

6 2010 S2 Q13
The continuous random variable $X$ has probability density function $f(x)$, where

$$f(x) = \begin{cases} a & \text{for } 0 \leq x < k \\ b & \text{for } k \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$E(X) = \frac{1 - 2b + ab}{2(a - b)}.$$ 

(ii) Show that the median, $M$, of $X$ is given by $M = \frac{1}{2a}$ if $a + b \geq 2ab$ and obtain an expression for the median if $a + b \leq 2ab$.

(iii) Show that $M < E(X)$.