

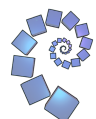
STEP Support Programme

STEP 3 Statistics Questions

1 2007 S3 Q13

A frog jumps towards a large pond. Each jump takes the frog either 1 m or 2 m nearer to the pond. The probability of a 1 m jump is p and the probability of a 2 m jump is q , where $p + q = 1$, the occurrence of long and short jumps being independent.

- (i) Let $p_n(j)$ be the probability that the frog, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, lands in the pond for the first time on its j th jump. Show that $p_2(2) = p$.
- (ii) Let u_n be the expected number of jumps, starting at a point $(n - \frac{1}{2})$ m away from the edge of the pond, required to land in the pond for the first time. Write down the value of u_1 . By finding first the relevant values of $p_n(m)$, calculate u_2 and show that $u_3 = 3 - 2q + q^2$.
- (iii) Given that u_n can be expressed in the form $u_n = A(-q)^{n-1} + B + Cn$, where A , B and C are constants (independent of n), show that $C = (1 + q)^{-1}$ and find A and B in terms of q . Hence show that, for large n , $u_n \approx \frac{n}{p + 2q}$ and explain carefully why this result is to be expected.



2 2010 S3 Q12

The infinite series S is given by

$$S = 1 + (1 + d)r + (1 + 2d)r^2 + \cdots + (1 + nd)r^n + \cdots ,$$

for $|r| < 1$. By considering $S - rS$, or otherwise, prove that

$$S = \frac{1}{1 - r} + \frac{rd}{(1 - r)^2} .$$

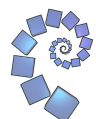
Arthur and Boadicea shoot arrows at a target. The probability that an arrow shot by Arthur hits the target is a ; the probability that an arrow shot by Boadicea hits the target is b . Each shot is independent of all others. Prove that the expected number of shots it takes Arthur to hit the target is $1/a$.

Arthur and Boadicea now have a contest. They take alternate shots, with Arthur going first. The winner is the one who hits the target first. The probability that Arthur wins the contest is α and the probability that Boadicea wins is β . Show that

$$\alpha = \frac{a}{1 - a'b'} ,$$

where $a' = 1 - a$ and $b' = 1 - b$, and find β .

Show that the expected number of shots in the contest is $\frac{\alpha}{a} + \frac{\beta}{b}$.



3 2013 S3 Q12

A list consists only of letters A and B arranged in a row. In the list, there are a letter A s and b letter B s, where $a \geq 2$ and $b \geq 2$, and $a + b = n$. Each possible ordering of the letters is equally probable. The random variable X_1 is defined by

$$X_1 = \begin{cases} 1 & \text{if the first letter in the row is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variables X_k ($2 \leq k \leq n$) are defined by

$$X_k = \begin{cases} 1 & \text{if the } (k-1)\text{th letter is } B \text{ and the } k\text{th is } A; \\ 0 & \text{otherwise.} \end{cases}$$

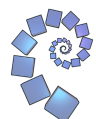
The random variable S is defined by $S = \sum_{i=1}^n X_i$.

- (i) Find expressions for $E(X_i)$, distinguishing between the cases $i = 1$ and $i \neq 1$, and show that $E(S) = \frac{a(b+1)}{n}$.
- (ii) Show that:

(a) for $j \geq 3$, $E(X_1 X_j) = \frac{a(a-1)b}{n(n-1)(n-2)}$;

(b) $\sum_{i=2}^{n-2} \left(\sum_{j=i+2}^n E(X_i X_j) \right) = \frac{a(a-1)b(b-1)}{2n(n-1)}$;

(c) $\text{Var}(S) = \frac{a(a-1)b(b+1)}{n^2(n-1)}$.



4 2005 S3 Q14

In this question, you may use the result

$$\int_0^{\infty} \frac{t^m}{(t+k)^{n+2}} dt = \frac{m!(n-m)!}{(n+1)!k^{n-m+1}},$$

where m and n are positive integers with $n \geq m$, and where $k > 0$.

The random variable V has density function

$$f(x) = \frac{C k^{a+1} x^a}{(x+k)^{2a+2}} \quad (0 \leq x < \infty),$$

where a is a positive integer. Show that $C = \frac{(2a+1)!}{a!a!}$.

Show, by means of a suitable substitution, that

$$\int_0^v \frac{x^a}{(x+k)^{2a+2}} dx = \int_{\frac{k^2}{v}}^{\infty} \frac{u^a}{(u+k)^{2a+2}} du$$

and deduce that the median value of V is k . Find the expected value of V .

The random variable V represents the speed of a randomly chosen gas molecule. The time taken for such a particle to travel a fixed distance s is given by the random variable $T = \frac{s}{V}$.

Show that

$$P(T < t) = \int_{\frac{s}{t}}^{\infty} \frac{C k^{a+1} x^a}{(x+k)^{2a+2}} dx \quad (*)$$

and hence find the density function of T . You may find it helpful to make the substitution $u = \frac{s}{x}$ in the integral (*).

Hence show that the product of the median time and the median speed is equal to the distance s , but that the product of the expected time and the expected speed is greater than s .

