

STEP Support Programme

STEP 2 Trigonometry Questions: Solutions

1 We have

$$\begin{aligned}
 \cos 3x &= \cos 2x \cos x - \sin 2x \sin x \\
 &= (2 \cos^2 x - 1) \cos x - (2 \sin x \cos x) \sin x \\
 &= 2 \cos^3 x - \cos x - 2 \cos x \sin^2 x \\
 &= 2 \cos^3 x - \cos x - 2 \cos x (1 - \cos^2 x) \\
 &= 2 \cos^3 x - \cos x - 2 \cos x + 2 \cos^3 x \\
 &= 4 \cos^3 x - 3 \cos x
 \end{aligned}$$

Since the answer is given, you do need to show every step. Remember “One equal sign per line, all equal signs aligned”!

Similarly, using $\sin 3x = \sin 2x \cos x + \cos 2x \sin x$ leads to $\sin 3x = 3 \sin x - 4 \sin^3 x$.

(i) Using $4 \sin^3 x = 3 \sin x - \sin 3x$ gives us:

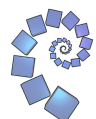
$$\begin{aligned}
 \int_0^\alpha (7 \sin x - 8 \sin^3 x) \, dx &= \int_0^\alpha (7 \sin x - 2(3 \sin x - \sin 3x)) \, dx \\
 &= \int_0^\alpha (\sin x + 2 \sin 3x) \, dx \\
 &= \left[-\cos x - \frac{2}{3} \cos 3x \right]_0^\alpha \\
 &= \left[-\cos \alpha - \frac{2}{3} \cos 3\alpha \right] - \left[-1 - \frac{2}{3} \right] \\
 &= -c - \frac{2}{3}(4c^3 - 3c) + \frac{5}{3} \\
 &= -\frac{8}{3}c^3 + c + \frac{5}{3}
 \end{aligned}$$

$I(\alpha) = 0$ when $c = 1$ (this can be easily found, as when $\alpha = 0$ the integral is of the form $\int_0^0 f(x) \, dx = 0$ giving $\cos(0) = 1$).

(ii) If we call Eustace’s attempt $J(\alpha)$ we have:

$$\begin{aligned}
 J(\alpha) &= \int_0^\alpha (7 \sin x - 8 \sin^3 x) \, dx \\
 &= \left[\frac{7}{2} \sin^2 x - 2 \sin^4 x \right]_0^\alpha \\
 &= \frac{7}{2} \sin^2 \alpha - 2 \sin^4 \alpha \\
 &= \frac{7}{2} (1 - \cos^2 \alpha) - 2(1 - \cos^2 \alpha)^2 \\
 &= -2c^4 + \frac{1}{2}c^2 + \frac{3}{2}
 \end{aligned}$$

You can then substitute $c = \cos(\beta) = -\frac{1}{6}$ and show that $I(\beta) = J(\beta)$, but this is a little fiddly (there are fractions with denominator 648 involved). Another way is to show that $6c + 1$ is a factor of the equation you get by solving $I(\alpha) = J(\alpha)$. Remember that for a “Show that” you must fully support your answer.



Equating $I(\alpha) = J(\alpha)$ gives:

$$\begin{aligned}
 -\frac{8}{3}c^3 + c + \frac{5}{3} &= -2c^4 + \frac{1}{2}c^2 + \frac{3}{2} \\
 12c^4 - 16c^3 - 3c^2 + 6c + 1 &= 0
 \end{aligned}
 \tag{*}$$

If you have already substituted $c = -\frac{1}{6}$ into $I(\alpha)$ and $J(\alpha)$ and shown that this is a solution then you can factorise out $(6c + 1)$ without further explanation of why you can do it.

Otherwise, you can:

- substitute $c = -\frac{1}{6}$ into (*) and show that this is a solution (this is not too bad, especially if you don't evaluate 6^3)
- use long division to show that $(6c + 1)$ is a factor (remember to show that the remainder is 0 in this case)!
- factorise out $(6c + 1)$ by inspection and then expand the brackets to show that this works.

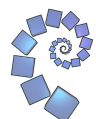
In each case you do need to end by stating something like "Hence $c = \cos(\beta) = -\frac{1}{6}$ gives Eustace the correct value of $I(\beta)$ ".

The equation fully factorises to give $(6c+1)(2c+1)(c-1)^2 = 0$, so we have $\cos \alpha = -\frac{1}{6}$, $\cos \alpha = -\frac{1}{2}$ and $\cos \alpha = 1$. There are no given restrictions on α , so we need the general solutions¹. These are:

$$\begin{aligned}
 \cos \alpha = 1 &\implies \alpha = 2n\pi \\
 \cos \alpha = -\frac{1}{2} &\implies \alpha = \frac{2}{3}\pi + 2n\pi \quad \text{or} \quad \alpha = -\frac{2}{3}\pi + 2n\pi \\
 \cos \alpha = -\frac{1}{6} &\implies \alpha = \cos^{-1}\left(-\frac{1}{6}\right) + 2n\pi \quad \text{or} \quad \alpha = -\cos^{-1}\left(-\frac{1}{6}\right) + 2n\pi
 \end{aligned}$$

Another method would be to show that $\alpha = 0 \implies c = 1$ is a solution first and then factorise out $(c - 1)$. You could keep factorising until you had fully factorised and then show that $c = -\frac{1}{6}$ is a solution.

¹See the topic notes.



2 There are two basic approaches for the first identity. Starting on the LHS gives:

$$\begin{aligned}
 \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) &\equiv \frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{x}{2}\right)} \\
 &\equiv \frac{1 - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{x}{2}\right)} \\
 &\equiv \frac{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)} \quad \text{Multiplying top and bottom by } \cos\left(\frac{x}{2}\right) \\
 &\equiv \frac{\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right]}{\left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right]} \\
 &\equiv \frac{1 - 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)} \\
 &\equiv \frac{1 - \sin x}{\cos x} \\
 &\equiv \sec x - \tan x
 \end{aligned}$$

Alternatively you can start on the RHS and use the $t = \tan \frac{1}{2}A$ formulae from the formula book (with $A = x$):

$$\begin{aligned}
 \sec x - \tan x &\equiv \frac{1 + t^2}{1 - t^2} - \frac{2t}{1 - t^2} \\
 &\equiv \frac{(1 - t)^2}{(1 - t)(1 + t)} \\
 &\equiv \frac{1 - t}{1 + t} \\
 &\equiv \frac{1 - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{x}{2}\right)} \\
 &\equiv \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)
 \end{aligned}$$

(i) Setting $x = \frac{\pi}{4}$ into the identity from the stem gives:

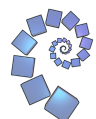
$$\tan\left(\frac{\pi}{4} - \frac{1}{2} \times \frac{\pi}{4}\right) = \sec\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right) = \sqrt{2} - 1.$$

Then note that $\frac{11}{24}\pi = \frac{1}{8}\pi + \frac{1}{3}\pi$ which gives:

$$\begin{aligned}
 \tan\left(\frac{1}{8}\pi + \frac{1}{3}\pi\right) &= \frac{\tan\left(\frac{1}{8}\pi\right) + \tan\left(\frac{1}{3}\pi\right)}{1 - \tan\left(\frac{1}{8}\pi\right)\tan\left(\frac{1}{3}\pi\right)} \\
 &= \frac{\sqrt{2} - 1 + \sqrt{3}}{1 - (\sqrt{2} - 1)\sqrt{3}} \\
 &= \frac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1} \quad \text{as required.}
 \end{aligned}$$

(ii) Since the answer is given here, you can multiply across and verify² that $(\sqrt{3} - \sqrt{6} + 1)(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}) = \sqrt{3} + \sqrt{2} - 1$.

²In this case verify means “expand the brackets and show it is true”. You MUST show all the working here as it is a “show that”. A table might be a nice clear way of showing your working.



Alternatively you can rationalise the denominator³:

$$\begin{aligned}
 \frac{\sqrt{3} + \sqrt{2} - 1}{\sqrt{3} - \sqrt{6} + 1} &= \frac{(\sqrt{3} + \sqrt{2} - 1)(\sqrt{3} + 1 + \sqrt{6})}{(\sqrt{3} + 1 - \sqrt{6})(\sqrt{3} + 1 + \sqrt{6})} \\
 &= \frac{2 + 4\sqrt{2} + 2\sqrt{3}}{2\sqrt{3} - 2} \\
 &= \frac{(1 + 2\sqrt{2} + \sqrt{3})(\sqrt{3} + 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} \\
 &= \frac{4 + 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{6}}{2} \\
 &= 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}
 \end{aligned}$$

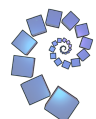
You should show a few more lines of working than presented here.

- (iii) Here using $x = \frac{11}{24}\pi$ gives us $\tan \frac{1}{48}\pi = \sec \frac{11}{24}\pi - \tan \frac{11}{24}\pi$. We know that $\tan \frac{11}{24}\pi = 2 + \sqrt{2} + \sqrt{3} + \sqrt{6}$ and we also have $\sec^2 \theta = 1 + \tan^2 \theta$. combining these gives us:

$$\tan \frac{1}{48}\pi = \sqrt{1 + \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)^2 - \left(2 + \sqrt{2} + \sqrt{3} + \sqrt{6}\right)}.$$

You then need to carefully expand the squared bracket (I would recommend a table for doing this) to get to the required result.

³If you had not been given the answer this is probably the approach you would have had to take. The first approach probably has less room for error though!



3 Using the given substitution we have:

$$\begin{aligned} \int \frac{1}{a^2 + a^2 \tan^2 \theta} \times a \sec^2 \theta \, d\theta &= \int \frac{1}{a} d\theta \\ &= \frac{1}{a} \times \theta + c \\ &= \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c. \end{aligned}$$

Since this result is in the “stem” of the question, you should expect to use it at least once (and probably more often) in the following question parts.

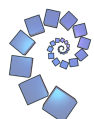
(i) (a) Using the substitution $t = \sin x$ gives us:

$$\begin{aligned} \int_0^{\frac{1}{2}\pi} \frac{\cos x}{1 + \sin^2 x} \, dx &= \int_0^1 \frac{\cos x}{1 + t^2} \frac{dt}{\cos x} \\ &= \int_0^1 \frac{1}{1 + t^2} \, dt \\ &= [\arctan t]_0^1 \quad \text{using the stem result} \\ &= \frac{\pi}{4} \end{aligned}$$

(b) Using the suggested substitution (which means that $\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{1}{2}x$) gives us:

$$\begin{aligned} \int_0^1 \frac{1 - t^2}{1 + 6t^2 + t^4} \, dt &= \int_0^{\frac{\pi}{2}} \frac{1 - \tan^2 \frac{1}{2}x}{1 + 6 \tan^2 \frac{1}{2}x + \tan^4 \frac{1}{2}x} \times \frac{1}{2} \sec^2 \frac{1}{2}x \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1 - \tan^2 \frac{1}{2}x}{(1 + 6 \tan^2 \frac{1}{2}x + \tan^4 \frac{1}{2}x) \cos^2 \frac{1}{2}x} \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1 - \frac{\sin^2 \frac{1}{2}x}{\cos^2 \frac{1}{2}x}}{\left(\cos^2 \frac{1}{2}x + 6 \sin^2 \frac{1}{2}x + \frac{\sin^4 \frac{1}{2}x}{\cos^2 \frac{1}{2}x}\right)} \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x}{(\cos^4 \frac{1}{2}x + 6 \sin^2 \frac{1}{2}x \cos^2 \frac{1}{2}x + \sin^4 \frac{1}{2}x)} \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x}{(\cos^2 \frac{1}{2}x + \sin^2 \frac{1}{2}x)^2 + 4 \sin^2 \frac{1}{2}x \cos^2 \frac{1}{2}x} \, dx \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} \, dx = \frac{1}{2} I \end{aligned}$$

The last step came from using $\cos 2A = \cos^2 A - \sin^2 A$, $\sin 2A = 2 \sin A \cos A$ and $\cos^2 A + \sin^2 A = 1$.



- (ii) Using the substitution $t = \tan \frac{1}{2}x$ in the same way as in part (i)(b) results in the integral $\frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + 3 \sin^2 x} dx$.

A further substitution of $y = \sqrt{3} \sin x$ gives the integral:

$$\frac{1}{2\sqrt{3}} \int_0^{\sqrt{3}} \frac{1}{1 + y^2} dy = \frac{1}{2\sqrt{3}} [\arctan y]_0^{\sqrt{3}}$$

which gives the final answer as $\frac{1}{2\sqrt{3}} \times \frac{\pi}{3} = \frac{1}{6\sqrt{3}}\pi$.

- 4 This is quite a long question (and in my opinion quite a hard one) as there is a lot going on.

- (i) Using $\cos \theta = \sin(90^\circ - \theta)$ gives us $\sin(90^\circ - \theta) = \sin 4\theta$. There is one obvious solution i.e. $90^\circ - \theta = 4\theta \implies \theta = 18^\circ$. However the question asks for **all** the values of θ , so there are almost certainly some more.

Using the periodicity of $\sin x$ we have:

$90^\circ - \theta = 4\theta + 360^\circ$	$\implies \theta = -54^\circ$	out of range
$90^\circ - \theta = 4\theta - 360^\circ$	$\implies \theta = 90^\circ$	OK
$90^\circ - \theta = 4\theta - 720^\circ$	$\implies \theta = 162^\circ$	OK
$90^\circ - \theta = 180^\circ - 4\theta$	$\implies \theta = 30^\circ$	OK
$90^\circ - \theta = 540^\circ - 4\theta$	$\implies \theta = 150^\circ$	OK

As is usually the case, a sketch helps make sure that you have all the possible values. You can also do this by solving $\cos \theta = \cos(90^\circ - 4\theta)$ and it might be a useful exercise to check that you get the same answers trying this method.

To find $\sin 18^\circ$ we use the double angle formulae to get:

$$\begin{aligned} \cos \theta &= 2 \sin 2\theta \cos 2\theta \\ \cos \theta &= 4 \sin \theta \cos \theta \times (1 - 2 \sin^2 \theta) . \end{aligned}$$

Then either $\cos \theta = 0$ (which is rejected as we are after $\theta = 18^\circ$) or $1 = 4 \sin \theta - 8 \sin^3 \theta$. It is usually helpful to write $\sin \theta = s$ so the equation is $8s^3 - 4s + 1 = 0$.

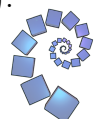
To solve this cubic, first note that $\theta = 30^\circ$ will be a solution (from above), so $s = \frac{1}{2}$ is a solution (which we will reject as we want $\theta = 18^\circ$). Using this to factorise we get:

$$8s^3 - 4s + 1 = (2s - 1)(4s^2 + 2s - 1) = 0 .$$

Using the quadratic formula gives the other two solutions as:

$$s = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4} .$$

The final step is to notice that $\sin 18^\circ$ will be positive, so $\sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1)$.



- (ii) Writing the equation in terms of $\sin x$ gives us:

$$\begin{aligned} 4 \sin^2 x + 1 &= 4 (2 \sin x \cos x)^2 \\ &= 16 \sin^2 x \cos^2 x \\ &= 16 \sin^2 x (1 - \sin^2 x) . \end{aligned}$$

Using $\sin x = s$ gives the quartic equation $16s^4 - 12s^2 + 1 = 0$. We can solve this for s^2 giving

$$s^2 = \frac{12 \pm \sqrt{144 - 4 \times 16}}{2 \times 16} = \frac{3 \pm \sqrt{9 - 4}}{8} = \frac{3 \pm \sqrt{5}}{8} .$$

This looks a little problematic, but we do know that we want $\sin x$ to be in the form $p + q\sqrt{5}$. You can solve $(p + q\sqrt{5})^2 = \frac{3 \pm \sqrt{5}}{8}$ but, on the assumption that the parts of the question are related, it might be worth considering $(\frac{1}{4}(\sqrt{5} - 1))^2$.

Noting that $(\frac{1}{4}(\sqrt{5} - 1))^2 = \frac{3 - \sqrt{5}}{8}$ means that we can deduce that the four values of $\sin x$ are $\pm \left(\frac{\sqrt{5} \pm 1}{4} \right)$.

- (iii) This is a “hence” question, so we need to refer to the previous parts (and trying to do it another way will probably gain no credit). If we take the equation from part (ii) and divide by 4 we get:

$$\sin^2 x + \frac{1}{4} = \sin^2 2x .$$

If we then take $3\alpha = x$ and $5\alpha = 30^\circ$ then this is the same as the equation in part (iii). $5\alpha = 30^\circ$ gives $\alpha = 6^\circ$ which gives $x = 18^\circ$ which is a solution to part (ii).

Finding the second solution is a little trickier. If we are to use the equation in part (ii) then we need $\sin 5\alpha = \pm \frac{1}{2}$. We also want probably want x to be “related” to 18° .

A bit of trial and error is needed, but if we take $x = 180^\circ + 18^\circ$ then we have $\sin x = -\frac{1}{4}(\sqrt{5} - 1)$, which is a solution to the equation in part (ii).

This means $\alpha = \frac{198^\circ}{3} = 66^\circ$ which gives $5\alpha = 330^\circ$ and so this satisfies $\sin 5\alpha = -\frac{1}{2}$. Hence a second solution is $\alpha = 66^\circ$.

