STEP Support Programme

STEP 2 Trigonometry Questions: Solutions

1 We have

$$\begin{aligned}\cos 3x &= \cos 2x \cos x - \sin 2x \sin x \\ &= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x \\ &= 2\cos^3 x - \cos x - 2\cos x \sin^2 x \\ &= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x) \\ &= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x \\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

Since the answer is given, you do need to show every step. Remember "One equal sign per line, all equal signs aligned"!

Similarly, using $\sin 3x = \sin 2x \cos x + \cos 2x \sin x$ leads to $\sin 3x = 3\sin x - 4\sin^3 x$.

(i) Using $4\sin^3 x = 3\sin x - \sin 3x$ gives us:

$$\begin{aligned}\int_0^\alpha \left(7\sin x - 8\sin^3 x\right) \mathrm{d}x &= \int_0^\alpha \left(7\sin x - 2(3\sin x - \sin 3x)\right) \mathrm{d}x \\ &= \int_0^\alpha (\sin x + 2\sin 3x)\,\mathrm{d}x \\ &= \left[-\cos x - \tfrac{2}{3}\cos 3x\right]_0^\alpha \\ &= \left[-\cos\alpha - \tfrac{2}{3}\cos 3\alpha\right] - \left[-1 - \tfrac{2}{3}\right] \\ &= -c - \tfrac{2}{3}(4c^3 - 3c) + \tfrac{5}{3} \\ &= -\tfrac{8}{3}c^3 + c + \tfrac{5}{3}\end{aligned}$$

$\mathrm{I}(\alpha) = 0$ when $c = 1$ (this can be easily found, as when $\alpha = 0$ the integral is of the form $\int_0^0 \mathrm{f}(x)\,\mathrm{d}x = 0$ giving $\cos(0) = 1$).

(ii) If we call Eustace's attempt $\mathrm{J}(\alpha)$ we have:

$$\begin{aligned}\mathrm{J}(\alpha) &= \int_0^\alpha \left(7\sin x - 8\sin^3 x\right) \mathrm{d}x \\ &= \left[\tfrac{7}{2}\sin^2 x - 2\sin^4 x\right]_0^\alpha \\ &= \tfrac{7}{2}\sin^2\alpha - 2\sin^4\alpha \\ &= \tfrac{7}{2}(1-\cos^2\alpha) - 2(1-\cos^2\alpha)^2 \\ &= -2c^4 + \tfrac{1}{2}c^2 + \tfrac{3}{2}\end{aligned}$$

You can then substitute $c = \cos(\beta) = -\frac{1}{6}$ and show that $\mathrm{I}(\beta) = \mathrm{J}(\beta)$, but this is a little fiddly (there are fractions with denominator 648 involved). Another way is to show that $6c+1$ is a factor of the equation you get by solving $\mathrm{I}(\alpha) = \mathrm{J}(\alpha)$. Remember that for a "Show that" you must fully support your answer.

Equating $\mathrm{I}(\alpha) = \mathrm{J}(\alpha)$ gives:

$$-\tfrac{8}{3}c^3 + c + \tfrac{5}{3} = -2c^4 + \tfrac{1}{2}c^2 + \tfrac{3}{2}$$
$$12c^4 - 16c^3 - 3c^2 + 6c + 1 = 0 \qquad (*)$$

If you have already substituted $c = -\frac{1}{6}$ into $\mathrm{I}(\alpha)$ and $\mathrm{J}(\alpha)$ and shown that this is a solution then you can factorise out $(6c + 1)$ without further explanation of why you can do it.

Otherwise, you can:

- substitute $c = -\frac{1}{6}$ into $(*)$ and show that this is a solution (this is not too bad, especially if you don't evaluate 6^3)
- use long division to show that $(6c + 1)$ is a factor (remember to show that the remainder is 0 in this case)!
- factorise out $(6c + 1)$ by inspection and then expand the brackets to show that this works.

In each case you do need to end by stating something like "Hence $c = \cos(\beta) = -\frac{1}{6}$ gives Eustace the correct value of $\mathrm{I}(\beta)$".

The equation fully factorises to give $(6c+1)(2c+1)(c-1)^2 = 0$, so we have $\cos\alpha = -\frac{1}{6}$, $\cos\alpha = -\frac{1}{2}$ and $\cos\alpha = 1$. There are no given restrictions on α, so we need the general solutions[1]. These are:

$$\begin{aligned}
\cos\alpha = 1 &\implies \alpha = 2n\pi \\
\cos\alpha = -\tfrac{1}{2} &\implies \alpha = \tfrac{2}{3}\pi + 2n\pi \quad \text{or} \quad \alpha = -\tfrac{2}{3}\pi + 2n\pi \\
\cos\alpha = -\tfrac{1}{6} &\implies \alpha = \cos^{-1}\left(-\tfrac{1}{6}\right) + 2n\pi \quad \text{or} \quad \alpha = -\cos^{-1}\left(-\tfrac{1}{6}\right) + 2n\pi
\end{aligned}$$

Another method would be to show that $\alpha = 0 \implies c = 1$ is a solution first and then factorise out $(c-1)$. You could keep factorising until you had fully factorised and then show that $c = -\frac{1}{6}$ is a solution.

[1] See the topic notes.

2 There are two basic approaches for the first identity. Starting on the LHS gives:

$$\begin{aligned}
\tan\left(\tfrac{\pi}{4}-\tfrac{x}{2}\right) &\equiv \frac{\tan\left(\frac{\pi}{4}\right)-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{x}{2}\right)}\\
&\equiv \frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\\
&\equiv \frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)} \qquad \textit{Multiplying top and bottom by } \cos\left(\tfrac{x}{2}\right)\\
&\equiv \frac{\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]}{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]}\\
&\equiv \frac{1-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\\
&\equiv \frac{1-\sin x}{\cos x}\\
&\equiv \sec x-\tan x
\end{aligned}$$

Alternatively you can start on the RHS and use the $t=\tan\frac{1}{2}A$ formulae from the formula book (with $A=x$):

$$\begin{aligned}
\sec x-\tan x &\equiv \frac{1+t^2}{1-t^2}-\frac{2t}{1-t^2}\\
&\equiv \frac{(1-t)^2}{(1-t)(1+t)}\\
&\equiv \frac{1-t}{1+t}\\
&\equiv \frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\\
&\equiv \tan\left(\tfrac{\pi}{4}-\tfrac{x}{2}\right)
\end{aligned}$$

(i) Setting $x=\frac{\pi}{4}$ into the identity from the stem gives:

$$\tan\left(\tfrac{\pi}{4}-\tfrac{1}{2}\times\tfrac{\pi}{4}\right)=\sec\left(\tfrac{\pi}{4}\right)-\tan\left(\tfrac{\pi}{4}\right)=\sqrt{2}-1\,.$$

Then note that $\frac{11}{24}\pi=\frac{1}{8}\pi+\frac{1}{3}\pi$ which gives:

$$\begin{aligned}
\tan\left(\tfrac{1}{8}\pi+\tfrac{1}{3}\pi\right) &= \frac{\tan\left(\frac{1}{8}\pi\right)+\tan\left(\frac{1}{3}\pi\right)}{1-\tan\left(\frac{1}{8}\pi\right)\tan\left(\frac{1}{3}\pi\right)}\\
&= \frac{\sqrt{2}-1+\sqrt{3}}{1-(\sqrt{2}-1)\sqrt{3}}\\
&= \frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1} \qquad \text{as required.}
\end{aligned}$$

(ii) Since the answer is given here, you can multiply across and verify[2] that $\left(\sqrt{3}-\sqrt{6}+1\right)\left(2+\sqrt{2}+\sqrt{3}+\sqrt{6}\right)=\sqrt{3}+\sqrt{2}-1$.

[2] In this case verify means "expand the brackets and show it is true". You MUST show all the working here as it is a "show that". A table might be a nice clear way of showing your working.

Alternatively you can rationalise the denominator[3]:

$$\begin{aligned}\frac{\sqrt{3}+\sqrt{2}-1}{\sqrt{3}-\sqrt{6}+1} &= \frac{(\sqrt{3}+\sqrt{2}-1)(\sqrt{3}+1+\sqrt{6})}{(\sqrt{3}+1-\sqrt{6})(\sqrt{3}+1+\sqrt{6})}\\ &= \frac{2+4\sqrt{2}+2\sqrt{3}}{2\sqrt{3}-2}\\ &= \frac{(1+2\sqrt{2}+\sqrt{3})(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}\\ &= \frac{4+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}}{2}\\ &= 2+\sqrt{2}+\sqrt{3}+\sqrt{6}\end{aligned}$$

You should show a few more lines of working than presented here.

(iii) Here using $x = \frac{11}{24}\pi$ gives us $\tan\frac{1}{48}\pi = \sec\frac{11}{24}\pi - \tan\frac{11}{24}\pi$. We know that $\tan\frac{11}{24}\pi = 2+\sqrt{2}+\sqrt{3}+\sqrt{6}$ and we also have $\sec^2\theta = 1+\tan^2\theta$. combining these gives us:

$$\tan\frac{1}{48}\pi = \sqrt{1+\left(2+\sqrt{2}+\sqrt{3}+\sqrt{6}\right)^2} - \left(2+\sqrt{2}+\sqrt{3}+\sqrt{6}\right)\,.$$

You then need to carefully expand the squared bracket (I would recommend a table for doing this) to get to the required result.

[3]If you had not been given the answer this is probably the approach you would have had to take. The first approach probably has less room for error though!

3 Using the given substitution we have:

$$\begin{aligned}\int \frac{1}{a^2 + a^2\tan^2\theta} \times a\sec^2\theta \,\mathrm{d}\theta &= \int \frac{1}{a}\mathrm{d}\theta \\ &= \frac{1}{a}\times\theta + c \\ &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c\,.\end{aligned}$$

Since this result is in the "stem" of the question, you should expect to use it at least once (and probably more often) in the following question parts.

(i) **(a)** Using the substitution $t = \sin x$ gives us:

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x &= \int_0^1 \frac{\cos x}{1+t^2}\,\frac{\mathrm{d}t}{\cos x} \\ &= \int_0^1 \frac{1}{1+t^2}\,\mathrm{d}t \\ &= \big[\arctan t\big]_0^1 \qquad \text{using the stem result} \\ &= \frac{\pi}{4}\end{aligned}$$

(b) Using the suggested substitution (which means that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\frac{1}{2}x$) gives us:

$$\begin{aligned}\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,\mathrm{d}t &= \int_0^{\frac{\pi}{2}} \frac{1-\tan^2\frac{1}{2}x}{1+6\tan^2\frac{1}{2}x+\tan^4\frac{1}{2}x} \times \tfrac{1}{2}\sec^2\tfrac{1}{2}x\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\tan^2\frac{1}{2}x}{\left(1+6\tan^2\frac{1}{2}x+\tan^4\frac{1}{2}x\right)\cos^2\frac{1}{2}x}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\dfrac{\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x}}{\left(\cos^2\frac{1}{2}x+6\sin^2\frac{1}{2}x+\dfrac{\sin^4\frac{1}{2}x}{\cos^2\frac{1}{2}x}\right)}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\left(\cos^4\frac{1}{2}x+6\sin^2\frac{1}{2}x\cos^2\frac{1}{2}x+\sin^4\frac{1}{2}x\right)}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\left(\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x\right)^2+4\sin^2\frac{1}{2}x\cos^2\frac{1}{2}x}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x = \tfrac{1}{2}I\end{aligned}$$

The last step came from using $\cos 2A = \cos^2 A - \sin^2 A$, $\sin 2A = 2\sin A\cos A$ and $\cos^2 A + \sin^2 A = 1$.

(ii) Using the substitution $t = \tan \frac{1}{2}x$ in the same way as in part **(i)(b)** results in the integral $\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+3\sin^2 x}\,\mathrm{d}x$.

A further substitution of $y = \sqrt{3}\sin x$ gives the integral:

$$\frac{1}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1+y^2}\,\mathrm{d}y = \frac{1}{2\sqrt{3}}\big[\arctan y\big]_0^{\sqrt{3}}$$

which gives the final answer as $\frac{1}{2\sqrt{3}} \times \frac{\pi}{3} = \frac{1}{6\sqrt{3}}\pi$.

4 This is quite a long question (and in my opinion quite a hard one) as there is a lot going on.

(i) Using $\cos\theta = \sin(90° - \theta)$ gives us $\sin(90° - \theta) = \sin 4\theta$. There is one obvious solution i.e. $90° - \theta = 4\theta \implies \theta = 18°$. However the question asks for **all** the values of θ, so there are almost certainly some more.

Using the periodicity of $\sin x$ we have:

$90° - \theta = 4\theta + 360°$	$\implies \theta = -54°$	out of range
$90° - \theta = 4\theta - 360°$	$\implies \theta = 90°$	OK
$90° - \theta = 4\theta - 720°$	$\implies \theta = 162°$	OK
$90° - \theta = 180° - 4\theta$	$\implies \theta = 30°$	OK
$90° - \theta = 540° - 4\theta$	$\implies \theta = 150°$	OK

As is usually the case, a sketch helps make sure that you have all the possible values. You can also do this by solving $\cos\theta = \cos(90° - 4\theta)$ and it might be a useful exercise to check that you get the same answers trying this method.

To find $\sin 18°$ we use the double angle formulae to get:

$$\cos\theta = 2\sin 2\theta\cos 2\theta$$
$$\cos\theta = 4\sin\theta\cos\theta \times \left(1 - 2\sin^2\theta\right).$$

Then either $\cos\theta = 0$ (which is rejected as we are after $\theta = 18°$) or $1 = 4\sin\theta - 8\sin^3\theta$. It is usually helpful to write $\sin\theta = s$ so the equation is $8s^3 - 4s + 1 = 0$.

To solve this cubic, first note that $\theta = 30°$ will be a solution (from above), so $s = \frac{1}{2}$ is a solution (which we will reject as we want $\theta = 18°$). Using this to factorise we get:

$$8s^3 - 4s + 1 = (2s - 1)(4s^2 + 2s - 1) = 0\,.$$

Using the quadratic formula gives the other two solutions as:

$$s = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1 \pm \sqrt{5}}{4}\,.$$

The final step is to notice that $\sin 18°$ will be positive, so $\sin 18° = \frac{1}{4}\left(\sqrt{5} - 1\right)$.

(ii) Writing the equation in terms of $\sin x$ gives us:

$$\begin{aligned} 4\sin^2 x + 1 &= 4\left(2\sin x\cos x\right)^2 \\ &= 16\sin^2 x\cos^2 x \\ &= 16\sin^2 x\left(1-\sin^2 x\right) \,. \end{aligned}$$

Using $\sin x = s$ gives the quartic equation $16s^4 - 12s^2 + 1 = 0$. We can solve this for s^2 giving

$$s^2 = \frac{12 \pm \sqrt{144 - 4\times 16}}{2\times 16} = \frac{3\pm\sqrt{9-4}}{8} = \frac{3\pm\sqrt{5}}{8}\,.$$

This looks a little problematic, but we do know that we want $\sin x$ to be in the form $p+q\sqrt{5}$. You can solve $\left(p+q\sqrt{5}\right)^2 = \dfrac{3\pm\sqrt{5}}{8}$ but, on the assumption that the parts of the question are related, it might be worth considering $\left(\frac{1}{4}(\sqrt{5}-1)\right)^2$.

Noting that $\left(\frac{1}{4}(\sqrt{5}-1)\right)^2 = \dfrac{3-\sqrt{5}}{8}$ means that we can deduce that the four values of $\sin x$ are $\pm\left(\dfrac{\sqrt{5}\pm 1}{4}\right)$.

(iii) This is a "hence" question, so we need to refer to the previous parts (and trying to do it another way will probably gain no credit). If we take the equation from part **(ii)** and divide by 4 we get:

$$\sin^2 x + \tfrac{1}{4} = \sin^2 2x\,.$$

If we then take $3\alpha = x$ and $5\alpha = 30^\circ$ then this is the same as the equation in part **(iii)**. $5\alpha = 30^\circ$ gives $\alpha = 6^\circ$ which gives $x = 18^\circ$ which is a solution to part **(ii)**.

Finding the second solution is a little trickier. If we are to use the equation in part **(ii)** then we need $\sin 5\alpha = \pm\frac{1}{2}$. We also want probably want x to be "related" to 18°.

A bit of trial and error is needed, but if we take $x = 180^\circ + 18^\circ$ then we have $\sin x = -\frac{1}{4}\left(\sqrt{5}-1\right)$, which is a solution to the equation in part **(ii)**.

This means $\alpha = \frac{198^\circ}{3} = 66^\circ$ which gives $5\alpha = 330^\circ$ and so this satisfies $\sin 5\alpha = -\frac{1}{2}$. Hence a second solution is $\alpha = 66^\circ$.