STEP Support Programme

STEP 3 Vectors Questions

1 SPECIMEN S2 Q9

(i) Let $\mathbf{a}$ and $\mathbf{b}$ be given vectors with $\mathbf{b} \neq \mathbf{0}$, and let $\mathbf{x}$ be a position vector. Find the condition for the sphere $|\mathbf{x}| = R$, where $R > 0$, and the plane $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ to intersect.

When this condition is satisfied, find the radius and the position vector of the centre of the circle in which the plane and sphere intersect.

(ii) Let $\mathbf{c}$ be a given vector, with $\mathbf{c} \neq \mathbf{0}$. The vector $\mathbf{x}'$ is related to the vector $\mathbf{x}$ by

$$\mathbf{x}' = \mathbf{x} - \frac{2(\mathbf{x} \cdot \mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}.$$

Interpret this relation geometrically.

2 92 S2 Q9

Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the position vectors of points A, B and C in three-dimensional space. Suppose that A, B, C and the origin O are not all in the same plane. Describe the locus of the point whose position vector $\mathbf{r}$ is given by

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

where λ and μ are scalar parameters. By writing this equation in the form $\mathbf{r} \cdot \mathbf{n} = p$ for a suitable vector $\mathbf{n}$ and scalar p, show that

$$-(\lambda + \mu)\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \lambda\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) + \mu\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 0$$

for all scalars λ, μ.

Deduce that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

Say briefly what happens if A, B, C and O are all in the same plane.

3 **93 S2 Q4**

Two non-parallel lines in 3-dimensional space are given by $\mathbf{r} = \mathbf{p}_1 + t_1\mathbf{m}_1$ and $\mathbf{r} = \mathbf{p}_2 + t_2\mathbf{m}_2$ respectively, where $\mathbf{m}_1$ and $\mathbf{m}_2$ are unit vectors. Explain by means of a sketch why the shortest distance between the two lines is

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2)\cdot(\mathbf{m}_1 \times \mathbf{m}_2)|}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}.$$

(i) Find the shortest distance between the lines in the case

$$\mathbf{p}_1 = (2, 1, -1) \qquad \mathbf{p}_2 = (1, 0, -2) \qquad \mathbf{m}_1 = \tfrac{1}{5}(4, 3, 0) \qquad \mathbf{m}_2 = \tfrac{1}{\sqrt{10}}(0, -3, 1).$$

(ii) Two aircraft, A_1 and A_2, are flying in the directions given by the unit vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ at constant speeds v_1 and v_2. At time $t = 0$ they pass the points $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. If d is the shortest distance between the two aircraft during the flight, show that

$$d^2 = \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2\,|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - [(\mathbf{p}_1 - \mathbf{p}_2)\cdot(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}.$$

(iii) Suppose that v_1 is fixed. The pilot of A_2 has chosen v_2 so that A_2 comes as close as possible to A_1. How close is that, if $\mathbf{p}_1, \mathbf{p}_2, \mathbf{m}_1$ and $\mathbf{m}_2$ are as in **(i)**?

4 **95 S3 Q8**

A plane π in 3-dimensional space is given by the vector equation $\mathbf{r}\cdot\mathbf{n} = p$, where $\mathbf{n}$ is a unit vector and p is a non-negative real number. If $\mathbf{x}$ is the position vector of a general point X, find the equation of the normal to π through X and the perpendicular distance of X from π.

The unit circles C_i, $i = 1, 2$, with centres $\mathbf{r}_i$, lie in the planes π_i given by $\mathbf{r}\cdot\mathbf{n}_i = p_i$, where the $\mathbf{n}_i$ are unit vectors, and p_i are non-negative real numbers. Prove that there is a sphere whose surface contains both circles only if there is a real number λ such that

$$\mathbf{r}_1 + \lambda\mathbf{n}_1 = \mathbf{r}_2 \pm \lambda\mathbf{n}_2.$$

Hence, or otherwise, deduce the necessary conditions that

$$(\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

and that

$$(p_1 - \mathbf{n}_1\cdot\mathbf{r}_2)^2 = (p_2 - \mathbf{n}_2\cdot\mathbf{r}_1)^2.$$

Interpret each of these two conditions geometrically.