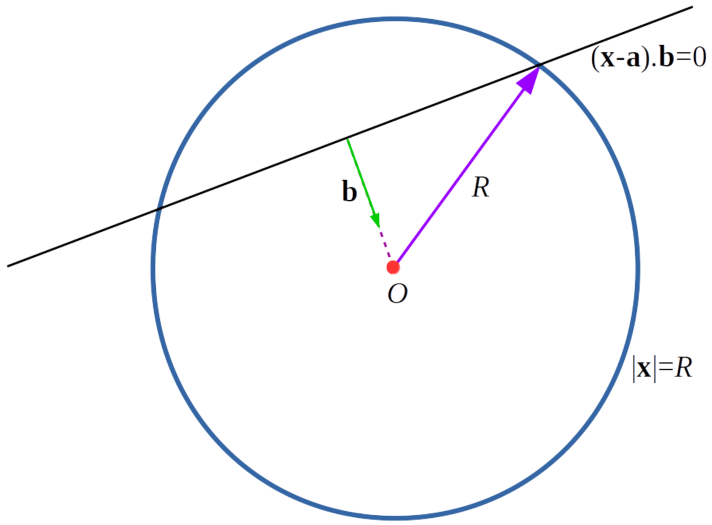
STEP Support Programme

STEP 3 Vectors Solutions

1 SPECIMEN S2 Q9

(i) A possible diagram could look like the following:



For the plane and sphere to intersect we need to have the perpendicular distance of the plane from the origin to be less than R.

The line that is perpendicular to the plane and which passes through the centre of the sphere (which is conveniently at the origin) has equation $\mathbf{x} = \lambda\mathbf{b}$. This line meets the plane when:

$$\begin{aligned}(\lambda\mathbf{b} - \mathbf{a}) \cdot \mathbf{b} &= 0\\ \lambda|\mathbf{b}|^2 &= \mathbf{a} \cdot \mathbf{b}\\ \lambda &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\end{aligned}$$

So the position vector of the point where this line meets the plane is:

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$$

For the plane to intersect the circle we need to have the distance from the origin to where the radius meets the plane to be less than R, i.e.

$$\left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}\right| < R$$
$$\frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|} < R$$

If this condition is satisfied then the centre of the circle where the plane and sphere intersect has position vector $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$ (from the above working).

The radius of the circle satisfies:

$$r^2 = R^2 - \left|\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}}\right|^2$$

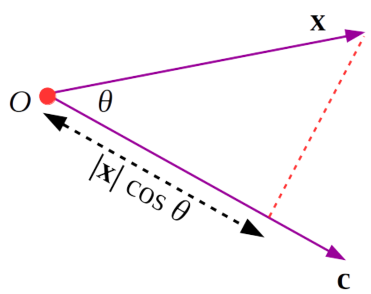
(ii) If you have met "projections" before you might be able to see what the relationship is rather more quickly than the method below.

First consider

$$\begin{aligned}\frac{(\mathbf{x}\cdot\mathbf{c})\mathbf{c}}{|\mathbf{c}|^2} &= \frac{|\mathbf{x}||\mathbf{c}|\cos\theta}{|\mathbf{c}|^2}\mathbf{c}\\ &= \frac{|\mathbf{x}|\cos\theta\mathbf{c}}{|\mathbf{c}|}\\ &= |\mathbf{x}|\cos\theta\hat{\mathbf{c}}\end{aligned}$$

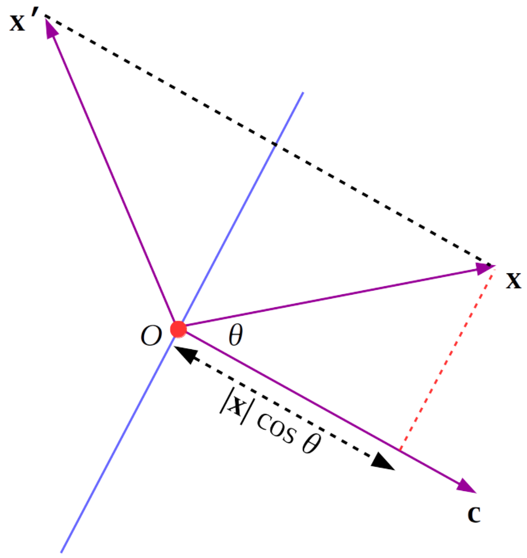
Where θ is the angle between $\mathbf{x}$ and $\mathbf{c}$ and $\hat{\mathbf{c}}$ is a unit vector in the direction of $\mathbf{c}$.

Drawing a picture of $\mathbf{x}$ and $\mathbf{c}$ gives a picture like this:



So $\dfrac{(\mathbf{x}\cdot\mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}$ is a vector of length $|\mathbf{x}|\cos\theta$ in the direction of $\mathbf{c}$. This is the *projection* of $\mathbf{x}$ onto vector $\mathbf{c}$.

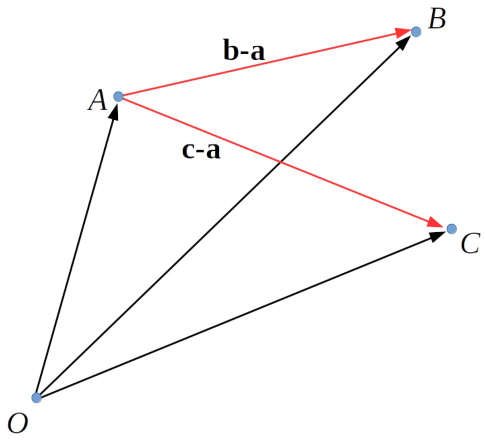
Using the above diagram as a starting point, we can add on vector $\mathbf{x}'$, where $\mathbf{x}' = \mathbf{x} - 2\times\dfrac{(\mathbf{x}\cdot\mathbf{c})\mathbf{c}}{|\mathbf{c}|^2}$. This gives us the following picture:



From this we can see that $\mathbf{x}'$ is the reflection of vector $\mathbf{x}$ in the plane through the origin with perpendicular vector $\mathbf{c}$.

2 92 S2 Q9

You can start by drawing a diagram — it's hard to draw in 3-D, however you can draw something like the following and imagine that the points A, B, C, O are the vertices of a tetrahedron.



Re-writing the given expression for $\mathbf{r}$ we have:

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) + \mu(\mathbf{c} - \mathbf{a})$$

Therefore $\mathbf{r}$ describes the points that lie in the plane which contains A, B and C. You can see this from the equation for $\mathbf{r}$ as this gives us that A is a point on the plane and that the two directions of the plane are given by $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$.

Since the vectors $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ describe the directions in the plane then the normal to the plane is given by $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$. This can be expanded to give:

$$\begin{aligned} (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) &= \mathbf{b} \times \mathbf{c} - \mathbf{a} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} \\ \implies \mathbf{n} &= \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} \end{aligned}$$

Remember that $\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$ and that $\mathbf{a} \times \mathbf{a} = 0$.

To find p, remember that $\mathbf{a}$ is a point in the plane and so we can substitute this into $\mathbf{r} \cdot \mathbf{n} = p$ to get:

$$\begin{aligned} \mathbf{r} \cdot \mathbf{n} &= p \\ \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) &= p \\ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= p \end{aligned}$$

Here we have used the fact that the dot (or scalar) product of two perpendicular vectors is 0.

We now have values for $\mathbf{n}$ and p. Using these and the general form of $\mathbf{r}$ given earlier we have:

$$\begin{aligned} \Big((1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}\Big) \cdot \Big(\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}\Big) &= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ (\cancel{1} - \lambda - \mu)\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \lambda\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) + \mu\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) &= \cancel{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} \\ -(\lambda + \mu)\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \lambda\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) + \mu\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) &= 0 \end{aligned}$$

as required.

For the "deduction" part we need to use the previous work. Setting $\lambda = 0$ gives us:

$$\begin{aligned} -\mu\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \mu\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) &= 0 \\ \implies \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \end{aligned}$$

and setting $\mu = 0$:

$$\begin{aligned} -\lambda\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) + \lambda\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) &= 0 \\ \implies \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) \end{aligned}$$

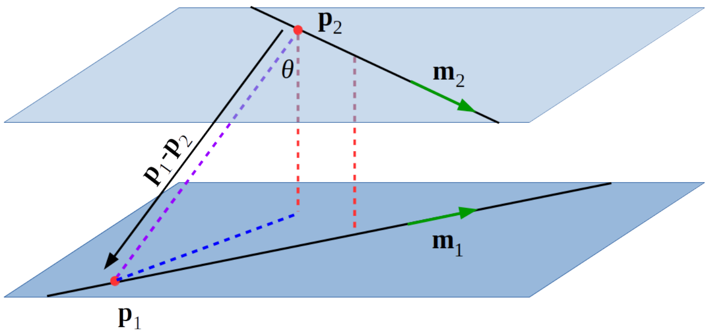
and so we have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$.

You could also have substituted $\lambda = -\mu$.

If A, B, C and O lie in the same plane then O is in the locus of $\mathbf{r}$. In this case we have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$.

3 93 S2 Q4

Draw a plane passing through $\mathbf{p}_1$ and which has direction vectors $\mathbf{m}_1$ and $\mathbf{m}_2$. This will contain the first line. Then draw another plane passing through $\mathbf{p}_2$ and which has direction vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ — this will contain the second line. The two planes will be parallel and the shortest distance between the two skew lines is the same as the distance between the planes (as shown by the red dotted lines).



The normal to the two planes is given by $\mathbf{m}_1 \times \mathbf{m}_2$, and a unit vector in this direction is $\dfrac{(\mathbf{m}_1 \times \mathbf{m}_2)}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}$.

To find the length of the dotted red line consider the triangle shown on the diagram above. The distance between the two planes is therefore $|\mathbf{p}_1 - \mathbf{p}_2| \cos\theta$. We also have:

$$|(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{m}_1 \times \mathbf{m}_2)| = |\mathbf{p}_1 - \mathbf{p}_2||(\mathbf{m}_1 \times \mathbf{m}_2)| \cos\theta$$

where we have taken the modulus sign as the angle between the two vectors may be the obtuse one (i.e. the angle between them might be $\pi - \theta$). Hence the required length is given by:

$$\frac{|(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{m}_1 \times \mathbf{m}_2)|}{|(\mathbf{m}_1 \times \mathbf{m}_2)|}$$

(i) We have:

$$\mathbf{m}_1 \times \mathbf{m}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{vmatrix}$$

or, more simply:

$$\mathbf{m}_1 \times \mathbf{m}_2 = \frac{1}{5\sqrt{10}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 0 \\ 0 & -3 & 1 \end{vmatrix} = \frac{1}{5\sqrt{10}} \begin{pmatrix} 3 \\ -4 \\ -12 \end{pmatrix}$$

and so:

$$|\mathbf{m}_1 \times \mathbf{m}_2| = \frac{1}{5\sqrt{10}} \sqrt{3^2 + (-4)^2 + (-12)^2} = \frac{13}{5\sqrt{10}}$$

We also have $\mathbf{p}_1 - \mathbf{p}_2 = -\mathbf{i} - \mathbf{j} - \mathbf{k}$ and so:

$$(\mathbf{p}_1 - \mathbf{p}_2) \cdot (\mathbf{m}_1 \times \mathbf{m}_2) = \frac{1}{5\sqrt{10}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ -12 \end{pmatrix} = \frac{13}{5\sqrt{10}}$$

hence the shortest distance between the two lines is 1.

(ii) The first time I did this part, I tried to use the "stem" result — which didn't help me much! The difference here is that the parameters are the same (rather than t_1 and t_2 as in the stem).

The position of the aircraft is given by:

$$\begin{aligned} A_1 &= \mathbf{p}_1 + tv_1\mathbf{m}_1 \\ A_2 &= \mathbf{p}_2 + tv_2\mathbf{m}_2 \end{aligned}$$

And so the distance between them is given by:

$$\mathbf{r} = \mathbf{p}_1 - \mathbf{p}_2 + t(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)$$

Hence we have:

$$\begin{aligned} |\mathbf{r}|^2 &= (\mathbf{p}_1 - \mathbf{p}_2 + t(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)) \cdot (\mathbf{p}_1 - \mathbf{p}_2 + t(v_1\mathbf{m}_1 - v_2\mathbf{m}_2)) \\ &= |\mathbf{p}_1 - \mathbf{p}_2|^2 + 2t(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2) + t^2|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 \end{aligned}$$

This is a quadratic in t^2. Differentiating tells us that the minimum distance occurs when $t = -\dfrac{(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}$, and so the minimum distance satisfies:

$$\begin{aligned} d^2 &= |\mathbf{p}_1 - \mathbf{p}_2|^2 + 2 \times -\frac{(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2} \times (\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2) + \\ &\qquad \left(-\frac{(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2}\right)^2 |v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 \\ &= |\mathbf{p}_1 - \mathbf{p}_2|^2 - 2\frac{\left[(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)\right]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2} + \frac{\left[(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)\right]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2} \\ &= |\mathbf{p}_1 - \mathbf{p}_2|^2 - \frac{\left[(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)\right]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2} \\ &= \frac{|\mathbf{p}_1 - \mathbf{p}_2|^2|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2 - \left[(\mathbf{p}_1 - \mathbf{p}_2) \cdot (v_1\mathbf{m}_1 - v_2\mathbf{m}_2)\right]^2}{|v_1\mathbf{m}_1 - v_2\mathbf{m}_2|^2} \end{aligned}$$

As required.

(iii) If the vectors are the same as in part **(i)**, then the closest distance between the *paths* of the two aircraft will be equal to 1. (Imagine the two contrails if it helps!). We have in this case $t_1 = v_1 t$ and $t_2 = v_2 t$. Say that the values of t_1 and t_2 when the two lines are closest together are $t_1 = T_1$ and $t_2 = T_2$

If v_1 is fixed, then we have the time when A_1 reaches the point where the paths are closest at time $T = \dfrac{T_1}{v_1}$. We want A_2 to reach the point on its path when the two paths are closest at the same time, so we need T to also satisfy $T = \dfrac{T_2}{v_2}$. Hence if we take $v_2 = \dfrac{T_2}{T_1} v_1$ then the closest the aircraft will be is the same as the closest that the two paths will be, i.e. is equal to 1.

I spent a long time trying to make this question work using the result from part **(ii)** before I realised we could use part **(i)**! There are probably more words and explanation here than you need to reproduce in an exam.

4 95 S3 Q8

The perpendicular line between $\mathbf{x}$ and plane π will have equation $\mathbf{x} + \lambda\mathbf{n}$. Where this meets the plane we have:

$$\begin{aligned}(\mathbf{x} + \lambda\mathbf{n}) \cdot \mathbf{n} &= p \\ \mathbf{x} \cdot \mathbf{n} + \lambda &= p \\ \lambda &= p - \mathbf{x} \cdot \mathbf{n}\end{aligned}$$

The distance between X and the plane will be given by $|\lambda\mathbf{n}| = |\lambda| = |p - \mathbf{x} \cdot \mathbf{n}|$. This could also be written as $|(\mathbf{r} - \mathbf{x}) \cdot \mathbf{n}|$ (where $\mathbf{r}$ is any point in the plane π).

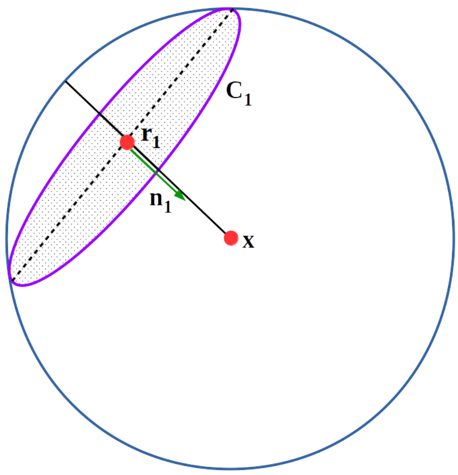
For the next request it is a good idea to unpick which direction the implication is in. You are asked to show:

"There is a sphere whose surface contains both circles **only if** there is a real number λ such that $\mathbf{r_1} + \lambda\mathbf{n_1} = \mathbf{r_2} \pm \lambda\mathbf{n_2}$"

This could be written as:

"**If** there is a sphere whose surface contains both circles **then** there is a real number λ such that $\mathbf{r_1} + \lambda\mathbf{n_1} = \mathbf{r_2} \pm \lambda\mathbf{n_2}$"

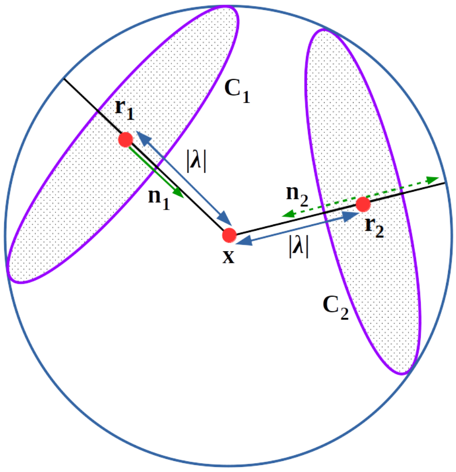
If there is to be a sphere containing circle C_1 then the line connecting the centre of the circle and the centre of the sphere will be perpendicular to the plane π_1 (i.e. the plane that circle C_1 lies in). Therefore this line must be parallel to $\mathbf{n_1}$. It is a good idea to try and draw a diagram to show this (tricky on a 2-D bit of paper!).



Note that I have shown vector $\mathbf{n_1}$ pointing towards the centre of the sphere - it could be pointing in the opposite direction.

The centre of the sphere must be along this line, so will have position vector $\mathbf{r_1} + \lambda\mathbf{n_1}$ for some value of λ. If $\mathbf{n_1}$ points towards the centre of the sphere then λ will be positive, and if $\mathbf{n_1}$ points in the opposite direction then λ will be negative. Since $\mathbf{n_1}$ is a unit vector the distance of the centre of the sphere from $\mathbf{r_1}$ (the centre of the circle) is equal to $|\lambda|$.

Now we assume that there is a sphere whose surface contains both the circles. Since both the circles have the same radius (they are both *unit circles*) the centres of the circles must be the same distance away from the centre of the sphere. A possible picture is shown below:



The position of the centre of the sphere will be given by $\mathbf{r_2} \pm \lambda \mathbf{n_2}$, where the sign will depend on the direction of $\mathbf{n_2}$ and sign of λ.

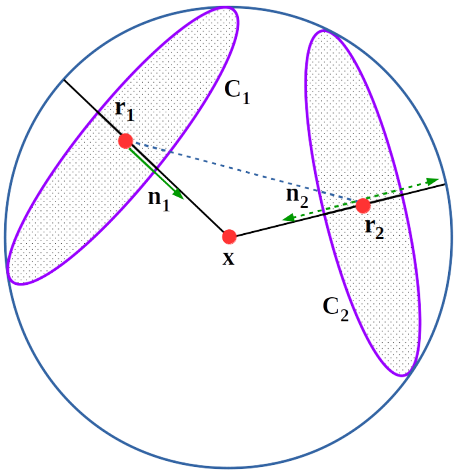
Equating the two expressions for the position vector of the centre of the sphere means that is there is a sphere whose surface contains the two unit circles then we have $\mathbf{r_1} + \lambda \mathbf{n_1} = \mathbf{r_2} \pm \lambda \mathbf{n_2}$.

From this we have:

$$\begin{aligned} \mathbf{r_1} - \mathbf{r_2} &= -\lambda(\mathbf{n_1} \mp \mathbf{n_2}) \\ (\mathbf{r_1} - \mathbf{r_2}) \cdot (\mathbf{n_1} \times \mathbf{n_2}) &= -\lambda(\mathbf{n_1} \mp \mathbf{n_2})(\mathbf{n_1} \times \mathbf{n_2}) \\ \therefore (\mathbf{r_1} - \mathbf{r_2}) \cdot (\mathbf{n_1} \times \mathbf{n_2}) &= 0 \end{aligned}$$

The last line here follows from the fact that $\mathbf{n_1} \times \mathbf{n_2}$ is perpendicular to both $\mathbf{n_1}$ and $\mathbf{n_2}$.

So if we have a sphere whose surface contains both the circles then we have $(\mathbf{r_1} - \mathbf{r_2}) \cdot (\mathbf{n_1} \times \mathbf{n_2}) = 0$. Geometrically this means that the line joining the centres of the two circles (shown as a blue dotted line below) lies in the plane which has directions given by $\mathbf{n_1}$ and $\mathbf{n_2}$. This plane also contains the centre of the sphere.



For the last part start with $\mathbf{r_1} + \lambda\mathbf{n_1} = \mathbf{r_2} + \lambda\mathbf{n_2}$. Then finding the scalar product of this with $\mathbf{n_1}$ and $\mathbf{n_2}$ in turn gives us:

$$\begin{aligned}
(\mathbf{r_1} + \lambda\mathbf{n_1}) \cdot \mathbf{n_1} = (\mathbf{r_2} + \lambda\mathbf{n_2}) \cdot \mathbf{n_1} \quad &\Longrightarrow \quad \mathbf{r_1} \cdot \mathbf{n_1} + \lambda = \mathbf{r_2} \cdot \mathbf{n_1} + \lambda\mathbf{n_2} \cdot \mathbf{n_1} \\
(\mathbf{r_1} + \lambda\mathbf{n_1}) \cdot \mathbf{n_2} = (\mathbf{r_2} + \lambda\mathbf{n_2}) \cdot \mathbf{n_2} \quad &\Longrightarrow \quad \mathbf{r_1} \cdot \mathbf{n_2} + \lambda\mathbf{n_1} \cdot \mathbf{n_2} = \mathbf{r_2} \cdot \mathbf{n_2} + \lambda
\end{aligned}$$

Rearranging both of these to get $\lambda\mathbf{n_1} \cdot \mathbf{n_2} - \lambda$ on one side and equating gives:

$$\begin{aligned}
&\mathbf{r_1} \cdot \mathbf{n_1} - \mathbf{r_2} \cdot \mathbf{n_1} = \mathbf{r_2} \cdot \mathbf{n_2} - \mathbf{r_1} \cdot \mathbf{n_2} \\
\Longrightarrow\ &p_1 - \mathbf{r_2} \cdot \mathbf{n_1} = p_2 - \mathbf{r_1} \cdot \mathbf{n_2}
\end{aligned}$$

Remember that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

Similarly, if you start with $\mathbf{r_1} + \lambda\mathbf{n_1} = \mathbf{r_2} - \lambda\mathbf{n_2}$ and follow the same steps you end up with $p_1 - \mathbf{r_2} \cdot \mathbf{n_1} = -p_2 + \mathbf{r_1} \cdot \mathbf{n_2} = -(p_2 - \mathbf{r_1} \cdot \mathbf{n_2})$.

hence if we square both sides we get:

$$(p_1 - \mathbf{r_2} \cdot \mathbf{n_1})^2 = (p_2 - \mathbf{r_1} \cdot \mathbf{n_2})^2$$

Using the expression for the distance of the point with position vector $\mathbf{x}$ from the plane $\mathbf{r}\cdot\mathbf{n} = p$ that we found at the start of the question, this tells us that the perpendicular distance of the centre of circle C_1 to the plane containing C_2 is the same as the the perpendicular distance of the centre of circle C_2 to the plane containing C_1. In the diagram below these distances are represented by the blue arrows (it's hard to draw this in 2-D!).