STEP Support Programme

STEP 3 Vectors Topic Notes

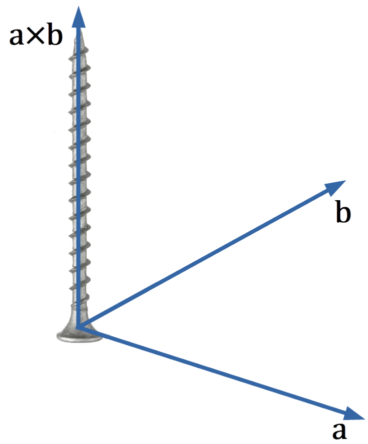
Vector Product

The vector product of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and θ is the angle between them.

The direction of $\hat{\mathbf{n}}$ is given by the *right hand rule* — if $\mathbf{a}$ is the thumb and $\mathbf{b}$ is the index finger then the direction of $\hat{\mathbf{n}}$ is given by the direction of the second finger[1]. Alternatively, you can consider a normal (right-handed) screw. If you turn the screw from $\mathbf{a}$ to $\mathbf{b}$ then the direction that the screw moves in is the direction of $\mathbf{a} \times \mathbf{b}$.



If you move your thumb to where the index finger was and the index finger to where the thumb was you should find that your second finger is now pointing in the opposite direction. With the screw analogy, if you turn in the opposite direction the screw will loosen and move downwards. This means that:

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Note that $\mathbf{a} \times \mathbf{a} = 0$ as in this case we have $\theta = 0$.

If $\mathbf{a} \times \mathbf{b} = 0$ then either $\mathbf{a} = 0$, or $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are parallel (i.e. one is a multiple of the other).

[1] This assumes that you do not have hyper-mobile joints or any broken fingers.

In determinant form we have:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{or} \quad \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix}$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

The *Scalar Triple Product* is given by:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and is unchanged by a circular shift of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. We also have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.

Vector equation of a line

Let $\mathbf{r}$ be a general point on a line that passes through point A with position vector $\mathbf{a}$ and which is parallel to the vector $\mathbf{b}$. Then we can write the equation of the line as:

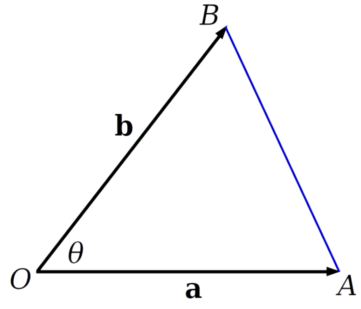
$$(\mathbf{r}-\mathbf{a})\times\mathbf{b} = 0 \quad \text{or} \quad \mathbf{r}\times\mathbf{b} = \mathbf{a}\times\mathbf{b}\,.$$

This is because the vector $\mathbf{r}-\mathbf{a}$ is parallel to $\mathbf{b}$, and the vector product of two parallel vectors is equal to 0.

Areas and Volumes

Area of a triangle

Consider a triangle with vertices O, A and B.



Let the position vector of point A be $\mathbf{a}$ and the position vector of point B be $\mathbf{b}$. Also let the angle between $\mathbf{a}$ and $\mathbf{b}$ be θ. The area of the triangle is given by:

$$\tfrac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

(Remembering that $\hat{\mathbf{n}}$ is a unit vector, so $|\hat{\mathbf{n}}| = 1$).

If the triangle had vertices A, B and C, then the vectors of the lengths relative to C are $\overrightarrow{CA} = \mathbf{a}-\mathbf{c}$ and $\overrightarrow{CB} = \mathbf{b}-\mathbf{c}$. If the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is θ then the area of the triangle is:

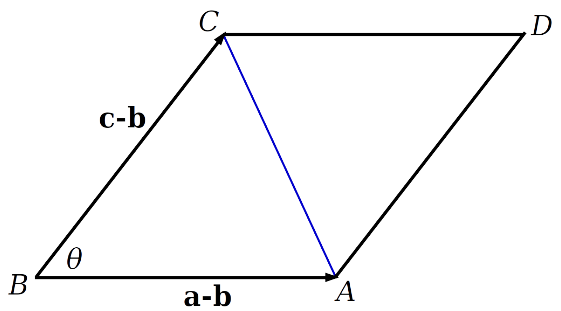
$$\tfrac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}|\sin\theta = \tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})|$$

Rearranging this gives:

$$\begin{aligned}\tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})| &= \tfrac{1}{2}|(\mathbf{a}\times\mathbf{b})-(\mathbf{c}\times\mathbf{b})-(\mathbf{a}\times\mathbf{c})+(\mathbf{c}\times\mathbf{c})| \\ &= \frac{1}{2}|(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})|\end{aligned}$$

Which is a nicely symmetrical result. You will not be expected to be able to quote this — however deriving this from $\tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})|$ would be fair game!

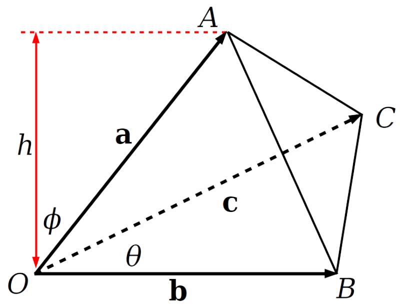
Area of a parallelogram Consider a parallelogram with vertices A, B, C and D.



The area of the parallelogram is twice the area of triangle ABC and so is equal to $||(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})|$.

Volume of a triangular based pyramid

Consider a pyramid with vertices O, A, B and C.



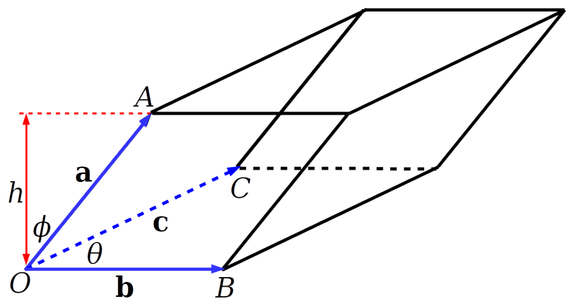
Let θ be the angle between OB and OC, and let ϕ be the angle between OA and the perpendicular height.

The volume of the pyramid is given by $\frac{1}{3} \times$ (area of the base) $\times\, h$. The area of the base is given by $\frac{1}{2}|\mathbf{b} \times \mathbf{c}|$. The height of the pyramid is given by $|\mathbf{a}|\cos\phi$. Using $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\phi$ we have:

$$\begin{aligned} \text{Volume } &= \tfrac{1}{3} \times |\mathbf{a}|\cos\phi \times \tfrac{1}{2}|\mathbf{b} \times \mathbf{c}| \\ &= \tfrac{1}{6}\,\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned}$$

Volume of a parallelepiped

The volume of a parallelepiped (shown below) is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. This can be derived in a similar way to the formula for the triangular based pyramid above.



Equation of a plane

- Vector $\mathbf{n}$ is perpendicular to plane Π, and plane Π passes through the point A with position vector $\mathbf{a}$. The equation of the plane can be written as $(\mathbf{r}-\mathbf{a})\cdot\mathbf{n} = 0$ or equivalently $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$.
- If we write $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ can be written as $ax + by + cz = d$ where $d = \mathbf{a} \cdot \mathbf{n}$.
- The equation of a plane Π can also be written as $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where $\mathbf{a}$ is the position vector of a point A in the plane, and $\mathbf{b}$ and $\mathbf{c}$ are two non-parallel vectors in the plane.

Other stuff on planes

- If two planes have normals $\mathbf{n}_1$ and $\mathbf{n}_2$ then the angle between the planes is given by:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

- The angle between a line with equation $\mathbf{r}_1 = \mathbf{a} + \lambda\mathbf{b}$ and the plane with equation $\mathbf{r}_2 \cdot \mathbf{n} = p$ is given by:

$$\cos\left(\frac{\pi}{2} - \theta\right) = \frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$
$$\implies \sin\theta = \frac{\mathbf{b} \cdot \mathbf{n}}{|\mathbf{b}||\mathbf{n}|}$$

  Note that the scalar product will involve the angle between the *normal* to the plane and the direction of the line. If the angle between the plane and the line is θ, then the angle between the normal to the plane and the line is $\frac{\pi}{2} - \theta$.

Intersection of a line and a plane

There are three possible situations here:

1 : There is a unique solution, so the line intersects the plane at a single point

2 : There are no solutions, i.e. the line never meets the plane — the line is parallel to the plane

3 : There are infinitely many solutions — in this case the line is contained in the plane

- **Example 1**

  Find the intersection(s) of the plane $\Pi : 2x + y - 2z = 4$ and the line $x - 1 = \dfrac{y-4}{2} = \dfrac{z+2}{3}$.

  Start by writing $x - 1 = \dfrac{y-4}{2} = \dfrac{z+2}{3} = \lambda$. We then have $x = \lambda + 1$, $y = 2\lambda + 4$ and $z = 3\lambda - 2$. Substituting these into the equation for the plane gives:

$$2(\lambda+1) + (2\lambda+4) - 2(3\lambda-2) = 4 \implies \lambda = 3$$

  And so the point of intersection is $(4, 10, 7)$.

  It is a very good idea to check these values in the original equations for the line and the plane — this point should satisfy both equations.

  If you try to find the points of intersection of the line equations $x - 1 = \dfrac{y-4}{2} = \dfrac{z+2}{2}$ and $x - 1 = \dfrac{y+2}{2} = \dfrac{z+2}{2}$ with the plane $\Pi : 2x + y - 2z = 4$ you should end up with $10 = 4$ in the first case (no solutions) and $4 = 4$ in the second (infinitely many solutions).

- **Example 2**
  Find any intersection(s) of the line $\mathbf{r} = \mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and the plane $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 10$.

  Substituting in the equation of the line gives:

$$\begin{aligned} \begin{pmatrix} 1+\lambda \\ -\lambda \\ 1+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} &= 10 \\ 3 + 3\lambda - \lambda - 1 - 2\lambda &= 10 \\ 2 &= 10 \end{aligned}$$

  So there are no points of intersection — the line is parallel to the plane.

Intersection of two planes

Here you can write both planes in Cartesian form, and then eliminate one of the variables.

Example:
Find the equations of the line of intersection of the planes $\Pi_1 : \mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 5$ and $\Pi_2 : \mathbf{r} \cdot (\mathbf{i} - 3\mathbf{j} + \mathbf{k}) = 7$.

We can write the equations of the planes as $2x + y - z = 5$ and $x - 3y + z = 7$. Eliminating z gives $3x - 2y = 12$ i.e. $y = \frac{3}{2}x - 6$. Substituting this into Π_1 gives $z = 2x + y - 5$ i.e. $z = \frac{7}{2}x - 11$.

Setting $x = \lambda$ gives the equation of the line as $\mathbf{r} = -6\mathbf{j} - 11\mathbf{k} + \lambda(\mathbf{i} + \frac{3}{2}\mathbf{j} + \frac{7}{2}\mathbf{k})$.

If the normals of the two plane equations are multiples of each other then the two planes are parallel and either never intersect, or are actually the same plane.

Perpendicular distances

- **Distance between two Parallel lines**
  If the two lines are parallel then they can be written as $\mathbf{r_1} = \mathbf{a_1} + \lambda\mathbf{b}$ and $\mathbf{r_2} = \mathbf{a_2} + \mu\mathbf{b}$. Therefore the vector between a general point on $\mathbf{r_1}$ and $\mathbf{r_2}$ can be written as $(\mathbf{a_1} - \mathbf{a_2}) + t\mathbf{b}$, and then this can then be minimised over t to find the shortest (perpendicular) distance.

  Example:
  Find the distance between the lines $\mathbf{r_1} = 2\mathbf{j} + 3\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r_2} = \mathbf{i} + \mathbf{k} + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$.

  We can write the vector of the line segment connecting two points on the lines as:

$$\begin{aligned} \begin{pmatrix} 2\lambda \\ 2-\lambda \\ 3+\lambda \end{pmatrix} - \begin{pmatrix} 1+2\mu \\ -\mu \\ 1+\mu \end{pmatrix} &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + (\lambda - \mu) \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \end{aligned}$$

So we want to minimise $\sqrt{(2t-1)^2+(2-t)^2+(t+2)^2} = \sqrt{6t^2-4t+9}$ which is minimised when $t = \frac{1}{3}$ (by differentiation of the quadratic). This gives the minimum/perpendicular distance as:

$$\sqrt{\frac{6}{9}-\frac{4}{3}+9} = \sqrt{\frac{25}{3}} = \frac{5\sqrt{3}}{3}$$

- **Distance between a point and a line**
  Use the same method as above!

  Example
  Find the distance between the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and the line with equation
  $\mathbf{r} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}-\mathbf{k})$.

  The vector of the line segment joining the point and a general point on the line is given by:

$$\begin{pmatrix}2\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\0\\-1\end{pmatrix}-\begin{pmatrix}1\\2\\-1\end{pmatrix}=\begin{pmatrix}1+2\lambda\\-3\\3-\lambda\end{pmatrix}$$

  So we want to minimise $\sqrt{(1+2\lambda)^2+(-3)^2+(3-\lambda)^2} = \sqrt{5\lambda^2-2\lambda+9}$. This is minimised when $\lambda = \frac{1}{5}$.

- **Distance between a point and a plane**
  If you know the normal to the plane, then the shortest distance to a point on the plane will be along the same direction as the normal.

  Example:
  Find the distance of the point $(1,3,-2)$ to the plane $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = 6$.

  The perpendicular line joining the point to the plane has equation:

$$\begin{pmatrix}1\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$$

  and where this meets the plane we have:

$$\begin{aligned}\begin{pmatrix}1+\lambda\\3+2\lambda\\-2+2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} &= 6\\(1+\lambda)+2(3+2\lambda)+2(-2+2\lambda) &= 6\\9\lambda+3 &= 6\\\lambda &= \frac{1}{3}\end{aligned}$$

  So the line meets the plane at the point:

$$\begin{pmatrix}1\\3\\-2\end{pmatrix}+\frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

  and this distance of this point from the point $(1,3,-2)$ is:

$$\frac{1}{3}\left|\begin{pmatrix}1\\2\\2\end{pmatrix}\right| = \frac{1}{3}\times 3 = 1$$

- **Distance between two Skew lines**

  If two lines are *skew* (that is they are not parallel and do not meet) then the shortest distance between them will be given by the line segment XY where X lies on one line, Y lies on the other and XY is perpendicular to both lines. If the two lines are $\mathbf{r_1} = \mathbf{a_1} + \lambda\mathbf{b_1}$ and $\mathbf{r_2} = \mathbf{a_2} + \lambda\mathbf{b_2}$ then a vector perpendicular to them both will be given by $\mathbf{b_1} \times \mathbf{b_2}$. You can then find a vector of the line segment between a general point on each line and show where this is parallel to $\mathbf{b_1} \times \mathbf{b_2}$.

  Example:
  Find the distance between the lines $\mathbf{r_1} = \mathbf{i}+\mathbf{j}+\lambda(2\mathbf{i}-\mathbf{j}+5\mathbf{k})$ and $\mathbf{r_2} = -\mathbf{i}+\mathbf{j}+2\mathbf{k}+\mu(2\mathbf{i}-5\mathbf{j}+\mathbf{k})$.

  A vector perpendicular to both lines is given by:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 5 \\ 2 & -5 & 1 \end{vmatrix} = \begin{pmatrix} 24 \\ 8 \\ -8 \end{pmatrix}$$

  So we want to find points X and Y on the two lines so that the line XY is parallel to $24\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}$ (or equivalently we could use $3\mathbf{i} + \mathbf{j} - \mathbf{k}$). Using the line equations we have:

$$\begin{pmatrix} 1+2\lambda \\ 1-\lambda \\ 5\lambda \end{pmatrix} - \begin{pmatrix} -1+2\mu \\ 1-5\mu \\ 2+\mu \end{pmatrix} = k\begin{pmatrix} 24 \\ 8 \\ -8 \end{pmatrix}$$

  This gives us three simultaneous equations:

$$2 + 2\lambda - 2\mu = 24k \qquad (1)$$
$$5\mu - \lambda = 8k \qquad (2)$$
$$5\lambda - \mu - 2 = -8k \qquad (3)$$

  Then $(1)+2(2)$ gives $2+8\mu = 40k \implies 4\mu+1 = 20k$ and $(3)+5(2)$ gives $24\mu-2 = 32k \implies 12\mu - 1 = 16k$. Substituting for 4μ gives $3(20k-1) - 1 = 16k \implies 44k = 4 \implies k = \frac{1}{11}$.

  The perpendicular distance between the two lines is therefore:

$$\left|\frac{1}{11}\begin{pmatrix} 24 \\ 8 \\ -8 \end{pmatrix}\right| = \frac{8}{11}\sqrt{3^2+1^2+1^2} = \frac{8\sqrt{11}}{11}$$

  There is a formula for the perpendicular distance between two skew lines — $\left|\dfrac{(\mathbf{a_1}-\mathbf{a_2})\cdot(\mathbf{b_1}\times\mathbf{b_2})}{|\mathbf{b_1}\times\mathbf{b_2}|}\right|$ — however I prefer to use the above method as it is easier to remember! In both this case and the method shown above if the lines actually meet then the methods will give a distance of 0 (and the first method can then be used to find the point of intersection).