# STEP Support Programme

## STEP 2 Vectors Questions: Solutions

It makes no real difference whether you choose to work with column vectors or **i**, **j**, **k** notation, choose whichever you prefer! I tend to find it easiest to write column vectors when working with pencil and paper, but **i**, **j**, **k** when writing in LaTeX.

**1** Let $m_3$ and $m_4$ have direction $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ (and hopefully there will be two possible directions of this form!). Since we know that the directions have to make angle $\pi/4$ with both $m_1$ and $m_2$, using the dot product we have:

$$a+b=\sqrt{2}\sqrt{a^2+b^2+c^2}\times\tfrac{1}{\sqrt{2}} \qquad \text{and}$$
$$a+c=\sqrt{2}\sqrt{a^2+b^2+c^2}\times\tfrac{1}{\sqrt{2}}\,.$$

Squaring both sides and simplifying gives $2ab=c^2$ and $2ac=b^2$. First thing to note is that $a\neq 0$, as if we have $a=0$ we would have $b=0$ and $c=0$ as well, which is not a very exciting direction. Since we are looking for direction vectors, WLOG[1] let $a=1$. We then have $2b=c^2$ and $2c=b^2$ which gives us the equation $b^4-8b=0$, which has solutions $b=0$ and $b=2$. If $b=0$ then $c=0$ and if $b=2$ then $c=2$.

The directions are therefore $\mathbf{i}$ and $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$. Using the dot product we find that the cosine of the angle between then is $\cos\theta=\frac{1}{3}$.

It is helpful to write down the positions of all the points (your $P$ and $Q$ might be the other way around):

$$A=(\mathbf{i}+\mathbf{j})\lambda$$
$$B=(\mathbf{i}+\mathbf{k})\lambda$$
$$P=\mathbf{i}$$
$$Q=\tfrac{1}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$$

**(i)** Use the above, we have:

$$\overrightarrow{AQ}=\left(\tfrac{1}{3}-\lambda\right)\mathbf{i}+\left(\tfrac{2}{3}-\lambda\right)\mathbf{j}+\tfrac{2}{3}\mathbf{k} \qquad \text{and}$$
$$\overrightarrow{BP}=(1-\lambda)\mathbf{i}-\lambda\mathbf{k}\,.$$

If they are to be perpendicular then we need $\overrightarrow{AQ}\cdot\overrightarrow{BP}=0$, and so we have $\left(\frac{1}{3}-\lambda\right)(1-\lambda)-\frac{2}{3}\lambda=0$. This simplifies to $3\lambda^2-6\lambda+1=0$ which has two solutions, $\lambda=1\pm\sqrt{\frac{2}{3}}$.

[1] Without loss of generality

**(ii)** Here we need the equations of the two lines:

$$r_{AQ} = (\mathbf{i}+\mathbf{j})\lambda + s\left[(1-3\lambda)\mathbf{i} + (2-3\lambda)\mathbf{j} + 2\mathbf{k}\right]$$
$$r_{BP} = \mathbf{i} + t\left[(1-\lambda)\mathbf{i} - \lambda\mathbf{k}\right] .$$

Your equations might look slightly different — for example if you started at point $Q$ rather than $A$ etc. For the direction of $r_{AQ}$ I have used $3\overrightarrow{AQ}$ so that there are no fractions.

Now we try and find any possible intersections of these two lines.

From the $\mathbf{j}$ component we have $\lambda + s(2-3\lambda) = 0$. Since $\lambda \neq 0$ and since we want a finite value of $s$ we have $s = \dfrac{\lambda}{3\lambda - 2}$ (where $\lambda \neq \frac{2}{3}$).

From the $\mathbf{k}$ component we have $2s = -\lambda t$, and so $t = \dfrac{2}{2-3\lambda}$.

Substituting these into the $\mathbf{i}$ component gives:

$$\lambda + \frac{\lambda}{3\lambda-2}(1-3\lambda) = 1 + \frac{2}{2-3\lambda}(1-\lambda)$$

Solving this equation gives $\lambda = \frac{2}{3}$, but this is not a possible value as $s$ and $t$ are undefined here. hence there are no non-zero values of $\lambda$ where the lines intersect.

**2** A clear diagram is invaluable here! Draw one (and the solution here will then be must easier to follow).

Since $C$ is the reflection of point $B$ in the line $OA$, then $OA$ is the perpendicular bisector of the line $BC$ and $OA$ bisects $\angle BOC$. This means that $A$ is on the diagonal $OA'$ of the rhombus $OBA'C$, and hence since $\mathbf{b} + \mathbf{c} = \mathbf{a}'$ we have $\mathbf{b} + \mathbf{c} = \lambda\mathbf{a}$ for some constant $\lambda$, and so $\mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$.

We also know that $BC$ is perpendicular to $OA$, which gives $(\mathbf{c} - \mathbf{b}) \cdot \mathbf{a} = 0$ and so $(\lambda\mathbf{a} - 2\mathbf{b}) \cdot \mathbf{a} = 0$ which gives the required result for $\lambda$.

Since point $D$ is the reflection of $C$ in the line $OB$ we can use our previous work to write down $\mathbf{d} = k\mathbf{b} - \mathbf{c}$ where $k = \dfrac{2\mathbf{b} \cdot \mathbf{c}}{\mathbf{b} \cdot \mathbf{b}}$. Using the earlier result for $\mathbf{c}$ we have $\mathbf{d} = (k+1)\mathbf{b} - \lambda\mathbf{a}$.

This means that $\mu = k + 1 = \dfrac{2\mathbf{b} \cdot \mathbf{c}}{\mathbf{b} \cdot \mathbf{b}} + 1$. Substituting for $\mathbf{c}$ gives:

$$\mu = \frac{2\mathbf{b} \cdot (\lambda\mathbf{a} - \mathbf{b})}{\mathbf{b} \cdot \mathbf{b}} + 1 = \frac{2\lambda\mathbf{b} \cdot \mathbf{a}}{\mathbf{b} \cdot \mathbf{b}} - 2\frac{\mathbf{b} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} + 1 = 4\frac{[\mathbf{a} \cdot \mathbf{b}]^2}{[\mathbf{a} \cdot \mathbf{a}][\mathbf{b} \cdot \mathbf{b}]} - 2 + 1\,. \qquad (*)$$

(Simplifying "$-2 + 1$" is left to you).

If $A$, $B$ and $D$ are collinear then $\overrightarrow{AB}$ is a multiple of $\overrightarrow{AD}$, i.e. $t\overrightarrow{AB} = \overrightarrow{AD}$ for some $t$ (you can write other equivalent statements such as $\overrightarrow{AD} = s\overrightarrow{DB}$).

$t\overrightarrow{AB} = \overrightarrow{AD}$ gives us $t(\mathbf{b} - \mathbf{a}) = \mu\mathbf{b} - (\lambda + 1)\mathbf{a}$. Equating coefficients of $\mathbf{a}$ and $\mathbf{b}$[2] gives us $t = \mu = \lambda + 1$, i.e. we have $\mu = \lambda + 1$.

If $\lambda = -\frac{1}{2}$ then $\mu = \frac{1}{2}$. Using the dot product formula gives cosine $\angle AOB$ as $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$, and we can use $(*)$ to get:

$$\mu = 4\cos^2\theta - 1$$

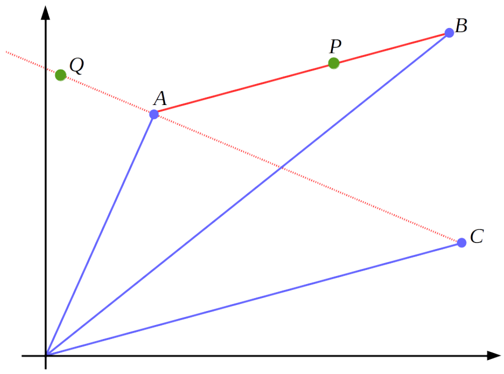
This gives $\cos^2\theta = \frac{3}{8}$, and as $\lambda$ has the same sign as $\mathbf{a} \cdot \mathbf{b}$ we have $\cos\theta = -\sqrt{\frac{3}{8}}$.

If $\lambda = -\frac{1}{2}$ then we have $\mathbf{d} = \frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{a}$ and so $D$ is the midpoint of $AB$.

[2] We can do this as $O$, $A$ and $B$ are not collinear, and so $\mathbf{a}$ and $\mathbf{b}$ are not multiples of each other.

**3** First thing to note is that the expressions for $\mathbf{p}$ and $\mathbf{q}$ have the standard form of a point on a line through two given points.

Point $P$ is on the line $AB$, and is strictly between $A$ and $B$. Point $Q$ is on the line $AC$ on the other side of $A$ to $C$ (so is closer to $A$ than $C$). Your diagram should include the lines $AB$ and $AC$, and look something like:




The point $P$ is situated so that $BP = \lambda AB$[3], which is more obvious if you rewrite $\mathbf{p}$ as $\mathbf{p} = \mathbf{b} + \lambda(\mathbf{a} - \mathbf{b})$ meaning that $\overrightarrow{OP} = \overrightarrow{OB} + \lambda\overrightarrow{BA}$. Similarly we have $CQ = \mu AC$. Therefore:

$$CQ \times BP = AB \times AC \implies \mu AC \times \lambda AB = AB \times AC$$

and so $\mu = \dfrac{1}{\lambda}$.

The equation of the line $PQ$ can be written as

$$\mathbf{r} = t\mathbf{p} + (1-t)\mathbf{q} = t\left[\lambda\mathbf{a} + (1-\lambda)\mathbf{b}\right] + (1-t)\left[\mu\mathbf{a} + (1-\mu)\mathbf{c}\right].$$

Substituting for $\mu$ and gathering terms gives:

$$\mathbf{r} = \left(t\lambda + \frac{1}{\lambda} - \frac{t}{\lambda}\right)\mathbf{a} + t(1-\lambda)\mathbf{b} + (1-t)\left(1 - \frac{1}{\lambda}\right)\mathbf{c}\,.$$

We then need the line to pass through $\mathbf{d} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$ (remember that $A$, $B$ and $C$ are non-collinear). First equate the coefficient of $\mathbf{b}$ in $\mathbf{r}$ to 1, which gives $t = \dfrac{1}{1-\lambda}$. Substituting into $\mathbf{r}$ gives:

$$\mathbf{r} = \left[\frac{1}{1-\lambda}\left(\lambda - \frac{1}{\lambda}\right) + \frac{1}{\lambda}\right]\mathbf{a} + \mathbf{b} + \left(1 - \frac{1}{1-\lambda}\right)\left(1 - \frac{1}{\lambda}\right)\mathbf{c}$$

which simplifies to $\mathbf{r} = -\mathbf{a} + \mathbf{b} + \mathbf{c}$ and hence the line passes through $D$.

For the last bit, note that $\mathbf{d} - \mathbf{c} = \mathbf{b} - \mathbf{a}$ and so $\overrightarrow{CD} = \overrightarrow{AB}$. This means that the sides $AB$ and $CD$ are equal and parallel, and so $ABDC$ is a parallelogram.

[3] Note that $AB$ represents the length between $A$ and $B$, not the vector which would be $\overrightarrow{AB}$.

**4** Using the dot product on $OA$ and $OB$ give us $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos 2\alpha$, where $\mathbf{a}$ is the position vector of $A$ etc. This gives $\cos 2\alpha = \frac{1}{3}$.

**(i)** If $L_1$ is equally inclined to both $OA$ and $OB$ then using the dot product gives:

$$\frac{m+n+p}{\left(\sqrt{m^2+n^2+p^2}\right)\times(\sqrt{3})} = \frac{5m-n-p}{\left(\sqrt{m^2+n^2+p^2}\right)\times(\sqrt{27})}.$$

Simplifying this gives the relationship $m = 2(n+p)$.

If $L_1$ is to be the angle bisector then, as well as satisfying the above, we need $L_1$ to be at an angle $\alpha$ with $OA$[4]. This means that we need:

$$\frac{m+n+p}{\left(\sqrt{m^2+n^2+p^2}\right)\times(\sqrt{3})} = \cos\alpha$$

and since $\cos 2\alpha = \frac{1}{3} = 2\cos^2\alpha - 1$ we have $\cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$. This gives us $(m+n+p)^2 = 2(m^2+n^2+p^2) \implies 2mn + 2np + 2pm = m^2+n^2+p^2$.

Substituting $m = 2(n+p)$ gives:

$$2n\times 2(n+p) + 2np + 2p\times 2(n+p) = 4(n+p)^2 + n^2 + p^2$$

which simplifies to $2np = n^2 + p^2$ i.e. $(n-p)^2 = 0$. Hence $n = p$ and $m = 4n$, so we can let $m\mathbf{i} + n\mathbf{j} + p\mathbf{k} = 4\mathbf{i} + \mathbf{j} + \mathbf{k}$ (or any other multiple of this).

**(ii)** Here we have $L_2$ inclined at an angle $\alpha$ to $OA$ (but not also to $OB$ which is what makes this different to part **(i)**). Hence we have the relationship $2uv+2vw+2wu = u^2+v^2+w^2$ (but we do not have $m = 2(n+p)$).

Comparing this to the equation $m^2+n^2+p^2 = 2(mn+np+pm)$ we can deduce that this represents all the lines at angle $\alpha = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$ to $OA$ — i.e. a double cone, vertex at $O$ with a central axis along the line containing $OA$.

[4]You could use the fact that $L_1$ has to be at an angle of $\alpha$ to $OB$, but $OA$ has simpler coefficients. If $L_1$ is the angle bisector then in lies in the plane containing $OA$ and $OB$.