STEP Support Programme

STEP 2 Vectors Topic Notes

Notation

Vectors can be written as column vectors, or by using **i**, **j**, **k** notation. We have:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Magnitude

The magnitude of a vector is the length of the vector. If $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{x}$ then:

$$|\mathbf{x}| = \sqrt{a^2 + b^2 + c^2}$$

Scalar product

The scalar (or dot) product of two vectors is given by:

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = ax + by + cz$$

We can express the magnitude using $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$.

The scalar product can also be expressed as:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between the two vectors.

Position vectors

Usually **a** is the position vector of point A with respect to a fixed point, usually the origin, O. We can write $\mathbf{a} = \overrightarrow{OA}$. To find the vector $\overrightarrow{AB}$ we can think about going backwards along OA and then along OB. This gives:

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$$

Distance between two points

The distance between points A and B can be found by using:

$$|\overrightarrow{AB}|^2 = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a})$$

Vector equations of lines

The vector equation of a line which passes through a point A with position vector $\mathbf{a}$ and travels in the direction described by vector $\mathbf{d}$ is:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$$

In particular, the vector equation of a line passing through A and B can be written as $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$, or by rearranging (and as is often used in STEP)

$$\mathbf{r} = (1 - \lambda)\mathbf{a} + \lambda\mathbf{b}$$

If $0 < \lambda < 1$ then the point on the line is between A and B, and if λ is outside this range then the point is on the line through A and B but not between them.

Two lines in **2 dimensions** either meet at a point, or are parallel, or are the same line.

In **3 dimensions** the two lines can meet at a point, be the same line, be parallel or they can be *skew* which is to say that they are not parallel, but also do not meet.

To show that two lines are parallel you need to show that the direction vectors are multiples of each other. If the direction vectors are multiples of each other and you can find a point in common then the two lines are the same line.

Finding the intersection of two lines can be done by equating components. Consider the lines $\mathbf{r_1} = \mathbf{a} + \lambda\mathbf{d}$ and $\mathbf{r_2} = \mathbf{b} + \mu\mathbf{m}$.

In the case where they are not parallel, where they meet we have:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

This gives three equations (one for the x component etc.). You can use two of these to find λ and μ, and then need to check to see if these satisfy the third equation. If they do satisfy all three equations then the lines meet at one point, otherwise the lines are skew.

Top Tips!

- Dot things together, and use the fact that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$
- $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
- $(\lambda\mathbf{a} + \mu\mathbf{b}) \cdot \mathbf{c} = \lambda(\mathbf{a} \cdot \mathbf{c}) + \mu(\mathbf{b} \cdot \mathbf{c})$
- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \geq 0$
- $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular or zero
- If $\mathbf{a}$ is a unit vector then $|\mathbf{a}| = 1$