

STEP Support Programme

STEP 2 Miscellaneous Questions: Solutions

1 The first 10 cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

(i) Substituting $y = k - x$ gives:

$$\begin{aligned}x^3 + (k - x)^3 &= kz^3 \\x^3 + k^3 - 3k^2x + 3kx^2 - x^3 &= kz^3 \\k^3 - 3k^2x + 3kx^2 - x^3 &= kz^3 \\k^3 - 3k^2x + 3kx^2 - x^3 &= kz^3 \quad \text{since } k > 0.\end{aligned}$$

Note that we need to state something like $k \neq 0$ before we divide by k .

We then have:

$$\begin{aligned}\frac{4z^3 - k^2}{3} &= \frac{4(k^2 - 3kx + 3x^2) - k^3}{3} \\&= \frac{3k^2 - 12kx + 12x^2}{3} \\&= k^2 - 4kx + 4x^2 \\&= (k - 2x)^2.\end{aligned}$$

You can if you like use $k = y + x$ to re-write the last line as $(y - x)^2$.

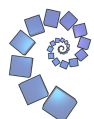
A perfect square is greater than or equal to 0, so we have $4z^3 - k^2 \geq 0 \implies z^3 \geq \frac{1}{4}k^2$. For the other part of the inequality use:

$$\begin{aligned}z^3 &= k^2 - 3kx + 3x^2 \\&= k^2 - 3x(k - x) \\&= k^2 - 3xy\end{aligned}$$

and as $x, y > 0$ we have $z^3 < k^2$. Therefore we have $\frac{1}{4}k^2 \leq z^3 < k^2$.

When $k = 20$ we have $100 \leq z^3 < 400$, so z must be 5, 6 or 7 (this is why you were asked to work out the first few cubes!). Testing each of these in $\frac{4z^3 - k^2}{3}$ shows that only $z = 7$ results in a perfect square. Using $\frac{4z^3 - k^2}{3} = 18^2 = (y - x)^2$ and $x + y = k = 20$, and assuming WLOG¹ that $y \geq x$, we can solve the simultaneous equations to get $x = 1$, $y = 19$. You can then check that $1^3 + 19^3 = 20 \times 7^3$ if you wish.

¹Without Loss Of Generality



(ii) Follow the same method as part (i)!

Substituting $y = z^2 - x$ gives:

$$\begin{aligned} x^3 + (z^2 - x)^3 &= kz^3 \\ x^3 + z^6 - 3z^4x + 3z^2x^2 - x^3 &= kz^3 \\ z^6 - 3z^4x + 3z^2x^2 &= kz^3 \quad \text{since } z > 0. \end{aligned}$$

We are then looking for something that will give us a perfect square. Comparison with part (i) leads us to:

$$\begin{aligned} \frac{4kz - z^4}{3} &= \frac{4(z^4 - 3z^2x + 3x^2) - z^4}{3} \\ &= z^4 - 4z^2x + 4x^2 \\ &= (z^2 - 2x)^2 \\ &= (y - x)^2. \end{aligned}$$

We then have $4kz - z^4 \geq 0 \implies z^3 \leq 4k$ (since $z > 0$ we can divide by z without changing the inequality direction).

We also have:

$$\begin{aligned} kz &= z^4 - 3z^2x + 3x^2 \\ &= z^4 - 3x(z^2 - x) \\ &= z^4 - 3xy \end{aligned}$$

and as $x, y > 0$ we have $kz < z^4 \implies k < z^3$. Hence we have $k < z^3 \leq 4k$. With $k = 19$ this gives $19 < z^3 \leq 76$ and so $z = 3$ or 4 . Both of these give perfect squares for $\frac{4kz - z^4}{3} = (y - x)^2$.

$z = 3$ gives $x + y = z^2 = 9$ and $(y - x)^2 = 49$, so $y = 8$ and $x = 1$.

$z = 4$ gives $x + y = z^2 = 16$ and $(y - x)^2 = 16$, so $y = 10$ and $x = 6$.

Again, you can check that these solve $x^3 + y^3 = kz^3$.



2 It is helpful to define a coordinate system for the tetrahedron. Let the line AB be on the x -axis so that the midpoint of AB is at the origin. This means that we have $A = (-\frac{1}{2}, 0, 0)$ and $B = (\frac{1}{2}, 0, 0)$. Using Pythagoras' theorem in $\triangle AOC$ gives $C = (0, \frac{\sqrt{3}}{2}, 0)$.

(i) The *centroid* of $\triangle ABC$ is a third of the distance from the centre of AB to C which gives $P = (0, \frac{\sqrt{3}}{6}, 0)$. You can use Pythagoras' theorem to find lengths PA and PD :

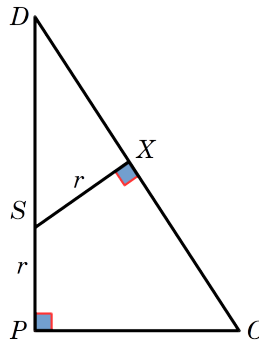
$$PA^2 = \frac{1}{4} + \frac{3}{36} = \frac{1}{3}$$

$$PD^2 = 1 - \frac{1}{3} = \frac{2}{3}$$

and hence $PD = \sqrt{\frac{2}{3}}$.

(ii) The angle between two adjacent faces is given by, e.g., $\angle DOC = \angle DOP$. Using the right-angled triangle $\triangle DOP$ gives $\cos(\angle DOP) = \frac{\frac{1}{6}\sqrt{3}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}$.

(iii) The centre of the sphere, S , must lie on the line DP by symmetry. Let X be where the sphere meets the face ABD .



We know that $OP = OX = \frac{\sqrt{3}}{6}$, $OD = \frac{\sqrt{3}}{2}$ and $DP = \frac{\sqrt{6}}{3}$. Using the right-angled triangle $\triangle DXS$ we have:

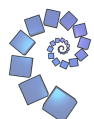
$$DS^2 = XS^2 + XD^2$$

$$\left(\frac{\sqrt{6}}{3} - r\right)^2 = r^2 + \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6}\right)^2$$

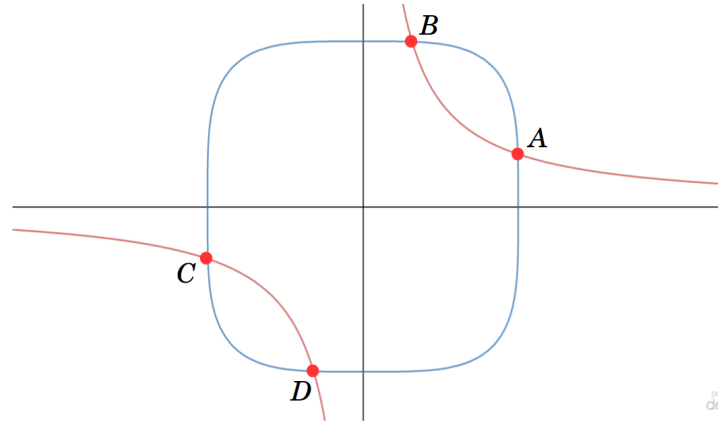
$$\frac{6}{9} - \frac{2\sqrt{6}}{3}r + r^2 = r^2 + \frac{1}{3}$$

$$\frac{2\sqrt{6}}{3}r = \frac{1}{3}$$

$$r = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12}$$



- 3** The curve $xy = v$ is symmetric in $y = x$ (which we can see as if we interchange x and y we get the same curve) and $y = -x$ (which we can see by using $x' = -y$ and $y' = -x$).
The curve $x^4 + y^4 = u$ is symmetric in $y = x$, $y = -x$, the x -axis (which can be seen by substituting $y' = -y$) and the y -axis (substitute $x' = -x$).



If $A = (\alpha, \beta)$ then $B = (\beta, \alpha)$, $C = (-\alpha, -\beta)$ and $D = (-\beta, -\alpha)$.

To show that $ABCD$ is a quadrilateral, you can show that AB is perpendicular to BC . We have:

$$m_{AB} \times m_{BC} = \frac{\beta - \alpha}{\alpha - \beta} \times \frac{\alpha + \beta}{\beta + \alpha} = -1.$$

Therefore AB and BC are perpendicular. You can then use the same technique to show that all of the four sides are perpendicular, or you can use symmetry in $y = x$ and $y = -x$ to show this.

To find the area:

$$\begin{aligned} AB^2 &= (\alpha - \beta)^2 + (\beta - \alpha)^2 = 2(\alpha - \beta)^2 \\ BC^2 &= (\beta + \alpha)^2 + (\alpha + \beta)^2 = 2(\alpha + \beta)^2. \end{aligned}$$

Remembering that $\alpha > \beta$ the area is:

$$AB \times BC = \sqrt{2}(\alpha - \beta) \times \sqrt{2}(\alpha + \beta) = 2(\alpha^2 - \beta^2).$$

The question asks us for the area in terms of u and v . Since α and β satisfy the equations of the curves we have $\alpha^4 + \beta^4 = u$ and $\alpha\beta = v$.

Considering $(\alpha^2 - \beta^2)^2$ gives us:

$$\begin{aligned} (\alpha^2 - \beta^2)^2 &= \alpha^4 - 2\alpha^2\beta^2 + \beta^4 \\ &= \alpha^4 + \beta^4 - 2(\alpha\beta)^2 \\ &= u - 2v^2 \end{aligned}$$

So the area is $2\sqrt{u - 2v^2}$.

Then for $u = 81$ and $v = 4$ the area is $2\sqrt{81 - 32} = 2\sqrt{49} = 14$.



4 We can deduce that

$$p(x) - 1 = (x - 1)^5 \times q(x) \quad (*)$$

where $q(x)$ is a quartic.

(i) Using $(*)$ we have $p(1) - 1 = (1 - 1)^5 \times q(1)$, so $p(1) - 1 = 0 \implies p(1) = 1$.

(ii) Differentiating $(*)$ gives:

$$\begin{aligned} p'(x) &= 5(x - 1)^4 \times q(x) + (x - 1)^5 \times q'(x) \\ &= (x - 1)^4 (5q(x) + (x - 1)q'(x)) \end{aligned}$$

and therefore $p'(x)$ is divisible by $(x - 1)^4$.

(iii) In a similar way to before we can write $p(x) + 1 = (x + 1)^5 \times q_2(x)$. Substituting in $x = -1$ gives $p(-1) = -1$ and differentiating can be used to show that $p'(x)$ is divisible by $(x + 1)^4$.

We can now write $p'(x) = k(x - 1)^4(x + 1)^4 = k(x^2 - 1)^4 = k(x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$.

Integrating gives:

$$p(x) = k\left(\frac{1}{9}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right) + c$$

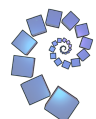
and using $p(1) = 1$ and $p(-1) = -1$ we have:

$$\begin{aligned} k\left(\frac{1}{9} - \frac{4}{7} + \frac{6}{5} - \frac{4}{3} + 1\right) + c &= 1 \\ k\left(-\frac{1}{9} + \frac{4}{7} - \frac{6}{5} + \frac{4}{3} - 1\right) + c &= -1 \end{aligned}$$

Adding these two together gives $c = 0$ and then the first one gives us:

$$k\left(\frac{35}{315} - \frac{180}{315} + \frac{378}{315} - \frac{420}{315} + \frac{315}{315}\right) = 1 \implies k = \frac{315}{128}.$$

So $p(x) = \frac{315}{128}\left(\frac{1}{9}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right)$.



5 $F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34$ and $F_{10} = 55$.

- (i) We have $F_i = F_{i-1} + F_{i-2}$ and, as long as $i \geq 4$, we have $F_{i-2} < F_{i-1}$ (when $i = 3$ we have equality as $F_1 = F_2$). Hence we have $F_i < 2F_{i-1}$ and so $\frac{1}{F_i} > \frac{1}{2F_{i-1}}$. We also have $\frac{1}{F_{i-1}} > \frac{1}{2F_{i-2}}$ (if $i \geq 5$) and so $\frac{1}{F_i} > \frac{1}{4F_{i-2}}$ etc.

We now have:

$$\begin{aligned} S &= \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots \\ S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{2F_3} + \frac{1}{2F_4} + \frac{1}{2F_5} + \dots \\ S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{2F_3} + \frac{1}{4F_3} + \frac{1}{8F_3} + \dots \\ S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \right) \end{aligned}$$

Using the values of F_1, F_2, F_3 and the sum of an infinite GP gives us:

$$S > 1 + 1 + \frac{1}{2} \times 2 = 3.$$

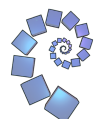
In a similar way to above we have $F_i > 2F_{i-2}$ (for $i \geq 4$) and so $\frac{1}{F_i} < \frac{1}{2F_{i-2}}$. This is slightly different to before as you have to split up odd and even terms.

$$\begin{aligned} S &= \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots \\ S &= \left(\frac{1}{F_1} + \frac{1}{F_2} \right) + \left(\frac{1}{F_4} + \frac{1}{F_6} + \frac{1}{F_8} + \dots \right) + \left(\frac{1}{F_3} + \frac{1}{F_5} + \frac{1}{F_7} + \dots \right) \\ S &< \left(\frac{1}{F_1} + \frac{1}{F_2} \right) + \left(\frac{1}{F_4} + \frac{1}{2F_4} + \frac{1}{4F_4} + \dots \right) + \left(\frac{1}{F_3} + \frac{1}{2F_3} + \frac{1}{4F_3} + \dots \right) \\ S &< 1 + 1 + \frac{1}{3} \times 2 + \frac{1}{2} \times 2 = 3\frac{2}{3} \end{aligned}$$

- (ii) For this part, use the same approach but take more terms before using the geometric series.

$$\begin{aligned} S &= \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots \\ S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) \\ S &> 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 = 1 + 1 + \frac{37}{30} \end{aligned}$$

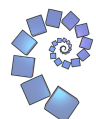
Hence $S > 3\frac{7}{30} > 3\frac{6}{30} = 3.2$



And for the upper limit:

$$\begin{aligned}
 S &= \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots \\
 S &= \left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} \right) + \left(\frac{1}{F_5} + \frac{1}{F_7} + \dots \right) + \left(\frac{1}{F_6} + \frac{1}{F_8} + \dots \right) \\
 S &< \left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} \right) + \frac{1}{F_5} \left(1 + \frac{1}{2} + \dots \right) + \frac{1}{F_6} \left(1 + \frac{1}{2} + \dots \right) \\
 S &< 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 + \frac{1}{8} \times 2
 \end{aligned}$$

and so $S < 3\frac{29}{60} < 3\frac{1}{2}$.



- 6 (i) Setting $a = 1$, $b = -1$, $x_n = x_{n+1} = x$ and $y_n = y_{n+1} = y$ gives the simultaneous equations:

$$\begin{aligned}x &= x^2 - y^2 + 1 \\ y &= 2xy + 1\end{aligned}$$

Using the second equation we have $x = \frac{y-1}{2y}$, which we can substitute into the first equation to get:

$$\begin{aligned}\frac{y-1}{2y} &= \left(\frac{y-1}{2y}\right)^2 - y^2 + 1 \\ 2y(y-1) &= (y-1)^2 - 4y^2 \times y^2 + 4y^2 \\ 2y^2 - 2y &= y^2 - 2y + 1 - 4y^4 + 4y^2 \\ 4y^4 - 3y^2 - 1 &= 0 \\ (y-1)(4y^3 + 4y^2 + y + 1) &= 0 \\ (y-1)(y+1)(4y^2 + 1) &= 0\end{aligned}$$

Therefore $y = 1$ or $y = -1$ ². Using $x = \frac{y-1}{2y}$ gives the values as $(x_1, y_1) = (0, 1)$ and $(1, -1)$.

- (ii) Taking $(x_1, y_1) = (-1, 1)$ gives $(x_2, y_2) = (a, b)$ and $(x_3, y_3) = (a^2 - b^2 + a, 2ab + b + 2)$. If the sequence is to have period 2 then we need $(x_1, y_1) = (x_3, y_3) \neq (x_2, y_2)$.

Using $(x_1, y_1) = (x_3, y_3)$ we have:

$$\begin{aligned}a^2 - b^2 + a &= -1 \\ 2ab + b + 2 &= 1\end{aligned}$$

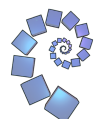
Similarly to before, we have $a = \frac{-b-1}{2b}$ and so:

$$\begin{aligned}\left(\frac{-b-1}{2b}\right)^2 - b^2 - \frac{b+1}{2b} &= -1 \\ (-b-1)^2 - 4b^4 - 2b(b+1) &= -4b^2 \\ b^2 + 2b + 1 - 4b^4 - 2b^2 - 2b &= -4b^2 \\ 4b^4 - 3b^2 - 1 &= 0\end{aligned}$$

This last equation is identical to the one in y for part (i), so we have $b = \pm 1$. This gives $(a, b) = (0, -1)$ or $(-1, 1)$, but since the second gives $(x_2, y_2) = (x_1, y_1)$ (i.e. sequence is constant, not period 2) we discard that one to leave us with just one solution, $(a, b) = (0, -1)$.

There is a neat solution here, where you can spot that if you let $a = -x$ and $b = -y$ you get the same equations as in part (i), which means you can deduce the values of a and b without solving the simultaneous equations.

²The equation $4y^2 = -1$ has no real solutions, and since we want (x, y) to be a point in the Cartesian plane, we want y to be real.



7 (i) The binomial expansion gives:

$$\begin{aligned} \left(1 + \frac{k}{100}\right)^{\frac{1}{2}} &= 1 + \frac{1}{2} \times \left(\frac{k}{100}\right) + \frac{1}{2!} \times \frac{1}{2} \times \frac{-1}{2} \times \left(\frac{k}{100}\right)^2 \\ &\quad + \frac{1}{3!} \times \frac{1}{2} \times \frac{-1}{2} \times \frac{-3}{2} \times \left(\frac{k}{100}\right)^3 + \dots \\ &\approx 1 + \frac{k}{200} - \frac{k^2}{80\,000} + \frac{k^3}{16\,000\,000} \end{aligned}$$

(a) Substituting $k = 8$ gives:

$$\begin{aligned} \left(1 + \frac{k}{100}\right)^{\frac{1}{2}} &= \left(\frac{108}{100}\right)^{\frac{1}{2}} \\ &= \left(\frac{3 \times 36}{100}\right)^{\frac{1}{2}} \\ &= \frac{6}{10} \times \sqrt{3} \end{aligned}$$

Using the binomial expansion (with $k = 8$) gives:

$$\begin{aligned} \frac{6}{10} \times \sqrt{3} &\approx 1 + \frac{8}{200} - \frac{8 \times 8}{80\,000} + \frac{8 \times 8 \times 8}{16\,000\,000} \\ &= 1 + \frac{4}{100} - \frac{8}{10\,000} + \frac{4 \times 8}{1\,000\,000} \\ &= 1.040032 - 0.0008 \\ &= 1.039232 \end{aligned}$$

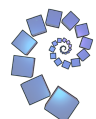
Then the approximation for $\sqrt{3}$ is given by:

$$\begin{aligned} \sqrt{3} &\approx \frac{1.039232}{0.6} \\ &= \frac{10.39232}{6} \end{aligned}$$

Carrying out the division gives $\sqrt{3} \approx 1.73205$.

(b) Here we need to find a suitable value of k , and remember that we want k to be small in order to get a good approximation. Comparing to the previous part, we want $100 + k = a^2 \times 6$. Starting with $a = 3$ we have:

$$\begin{aligned} 3^2 \times 6 = 54 &\implies k = -46 \\ 4^2 \times 6 = 96 &\implies k = -4 \\ 5^2 \times 6 = 150 &\implies k = 50 \end{aligned}$$



$k = 50$ is not a great choice as it is not small, so take $k = -4$.
Substituting $k = -4$ gives:

$$\begin{aligned} \left(\frac{96}{100}\right)^{\frac{1}{2}} &= \frac{4}{10} \times \sqrt{6} \\ &\approx 1 - \frac{4}{200} - \frac{4 \times 4}{80\,000} - \frac{4 \times 4 \times 4}{16\,000\,000} \\ &= 1 - \frac{2}{100} - \frac{2}{10\,000} - \frac{4}{1\,000\,000} \\ &= 1 - 0.020204 \\ &= 0.979796 \end{aligned}$$

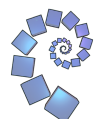
And so we have $\sqrt{6} \approx 9.79796 \div 4$, i.e. $\sqrt{6} \approx 2.44949$.

(ii) The first two terms of the binomial expansion of $\left(1 + \frac{k}{1000}\right)^{\frac{1}{3}}$ gives us:

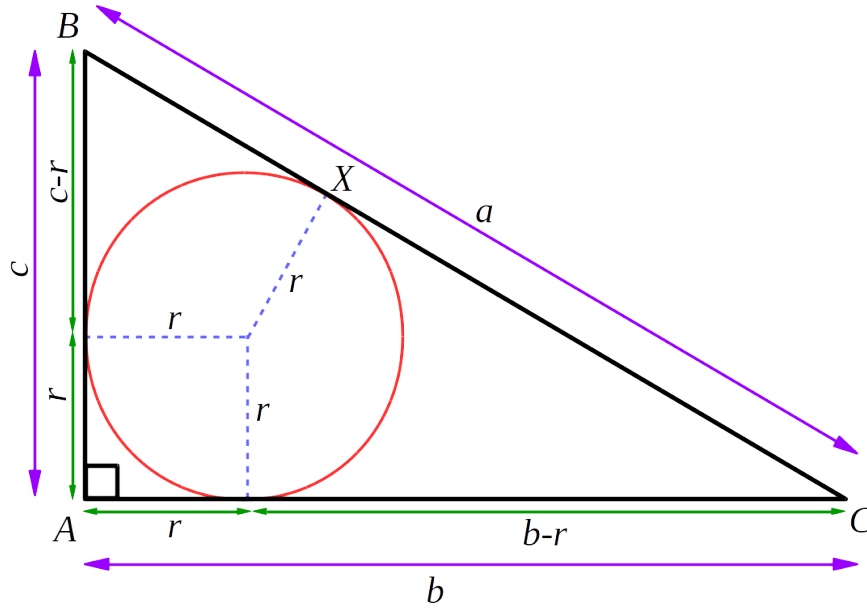
$$\left(1 + \frac{k}{1000}\right)^{\frac{1}{3}} \approx 1 + \frac{k}{3000}.$$

By comparing to the previous part, we want to find a value of k so that $1000+k = a^3 \times 3$, where k is small compared to 1000. The value of a which gives the smallest k is $a = 7$, which gives $1000 + k = 1029 \implies k = 29$. We then have:

$$\begin{aligned} \left(\frac{1029}{1000}\right)^{\frac{1}{3}} &\approx 1 + \frac{29}{3000} \\ \frac{7}{10} \times \sqrt[3]{3} &\approx \frac{3029}{3000} \\ \sqrt[3]{3} &\approx \frac{3029}{3000} \times \frac{10}{7} && \text{and so} \\ \sqrt[3]{3} &\approx \frac{3029}{2100} && \text{as required.} \end{aligned}$$

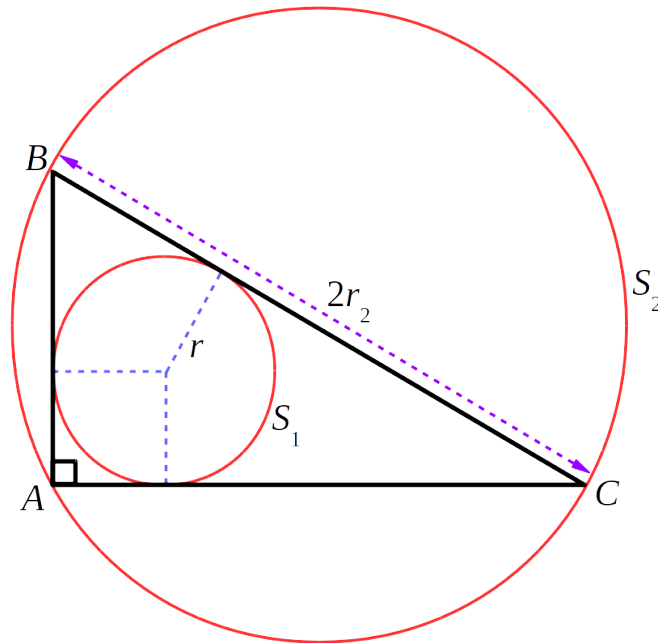


8 You should start by drawing a large and clear diagram, maybe something like below:



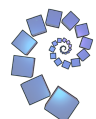
Then we have $CX = b - r$ and $BX = c - r$ (adjacent sides of a kite). Hence $a = (b - r) + (c - r)$ and so $2r = b + c - a$.

We now have another diagram with the **circumcircle** shown as well as the **incircle**.



Since $\triangle ABC$ is right-angled, BC is a diameter of S_2 . Hence the radius of S_2 satisfies $2r_2 = a$. The area of S_2 is πr_2^2 and the area between S_1 and the triangle is $\frac{1}{2}bc - \pi r^2$. Using the given fact about the ratio of these we have:

$$R\pi r_2^2 = \frac{1}{2}bc - \pi r^2$$



The result we are trying to obtain does not contain r or r_2 , so we substitute for these.

$$\begin{aligned} R\pi \left(\frac{a}{2}\right)^2 &= \frac{1}{2}bc - \pi \left(\frac{b+c-a}{2}\right)^2 \\ R\pi &= \frac{2bc}{a^2} - \pi \left(\frac{b+c-a}{a}\right)^2 \\ R\pi &= \frac{2bc}{a^2} - \pi(q-1)^2 \end{aligned}$$

This is starting to look promising, but the $\frac{2bc}{a^2}$ needs writing in terms of q . We have:

$$\begin{aligned} q^2 &= \left(\frac{b+c}{a}\right)^2 \\ &= \frac{b^2 + c^2 + 2bc}{a^2} \\ &= \frac{a^2 + 2bc}{a^2} \quad \text{using Pythagoras' theorem} \\ &= 1 + \frac{2bc}{a^2} \end{aligned}$$

So we now have:

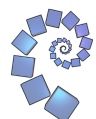
$$\begin{aligned} R\pi &= q^2 - 1 - \pi(q-1)^2 \\ &= q^2 - 1 - \pi(q^2 - 2q + 1) \\ &= q^2 - 1 - \pi q^2 + 2\pi q - \pi \\ &= -(\pi-1)q^2 + 2\pi q - (\pi+1) \quad \text{as required} \end{aligned}$$

Note that this is a quadratic in q , and will have a maximum when $\frac{d}{dq}(\pi R) = 0$, i.e. when

$$q = \frac{2\pi}{2(\pi-1)} = \frac{\pi}{\pi-1}. \quad \text{This gives}$$

$$\begin{aligned} \pi R_{\max} &= -(\pi-1) \left(\frac{\pi}{\pi-1}\right)^2 + 2\pi \left(\frac{\pi}{\pi-1}\right) - (\pi+1) \\ &= -\frac{\pi^2}{\pi-1} + \frac{2\pi^2}{\pi-1} - (\pi+1) \\ &= \frac{\pi^2}{\pi-1} - \frac{(\pi+1)(\pi-1)}{\pi-1} \\ &= \frac{\pi^2 - \pi^2 + 1}{\pi-1} \\ &= \frac{1}{\pi-1} \end{aligned}$$

And so since $R_{\max} = \frac{1}{\pi(\pi-1)}$ we have $R \leq \frac{1}{\pi(\pi-1)}$.



- 9 (i) One stumbling block is not reading all the information in the “stem”! Since you are told that $\lambda = 1 + \sqrt{2}$ you know that $\lambda - 1 = \sqrt{2}$ etc.

We have:

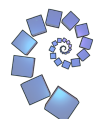
$$\begin{aligned}
 \sum_{r=0}^n b_r &= (\lambda^0 - \mu^0) + (\lambda^1 - \mu^1) + (\lambda^2 - \mu^2) + \dots + (\lambda^n - \mu^n) \\
 &= (1 + \lambda^1 + \lambda^2 + \dots + \lambda^n) - (1 + \mu^1 + \mu^2 + \dots + \mu^n) \\
 &= \frac{\lambda^{n+1} - 1}{\lambda - 1} - \frac{\mu^{n+1} - 1}{\mu - 1} \\
 &= \frac{\lambda^{n+1} - 1}{\sqrt{2}} - \frac{\mu^{n+1} - 1}{-\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}} (\lambda^{n+1} + \mu^{n+1}) - 2 \times \frac{1}{\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}} a_{n+1} - \sqrt{2}
 \end{aligned}$$

Similarly:

$$\begin{aligned}
 \sum_{r=0}^n a_r &= (\lambda^0 + \mu^0) + (\lambda^1 + \mu^1) + (\lambda^2 + \mu^2) + \dots + (\lambda^n + \mu^n) \\
 &= (1 + \lambda^1 + \lambda^2 + \dots + \lambda^n) + (1 + \mu^1 + \mu^2 + \dots + \mu^n) \\
 &= \frac{\lambda^{n+1} - 1}{\lambda - 1} + \frac{\mu^{n+1} - 1}{\mu - 1} \\
 &= \frac{\lambda^{n+1} - 1}{\sqrt{2}} + \frac{\mu^{n+1} - 1}{-\sqrt{2}} \\
 &= \frac{1}{\sqrt{2}} (\lambda^{n+1} - \mu^{n+1}) \\
 &= \frac{1}{\sqrt{2}} b_{n+1}
 \end{aligned}$$

- (ii) Here we have a “nested sum”. Start by evaluating the “inner sum”.

$$\begin{aligned}
 \sum_{m=0}^{2n} \left(\sum_{r=0}^m a_r \right) &= \sum_{m=0}^{2n} \left(\frac{1}{\sqrt{2}} b_{m+1} \right) \\
 &= \sum_{m=0}^{2n+1} \left(\frac{1}{\sqrt{2}} b_m \right) \quad \text{since } b_0 = 0 \\
 &= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}} a_{(2n+1)+1} - \sqrt{2} \right) \\
 &= \frac{1}{2} (a_{2n+2} - 2) \\
 &= \frac{1}{2} (\lambda^{2n+2} + \mu^{2n+2} - 2)
 \end{aligned}$$



We are trying to get something like $(b_{n+1})^2$.³ Noting that $\lambda \times \mu = 1 - 2 = -1$, and using the fact that n is odd, so that $n + 1$ is even and $(\lambda\mu)^{n+1} = 1$ we have:

$$\begin{aligned} \sum_{m=0}^{2n} \left(\sum_{r=0}^m a_r \right) &= \frac{1}{2} (\lambda^{2n+2} + \mu^{2n+2} - 2) \\ &= \frac{1}{2} \left((\lambda^{n+1})^2 + (\mu^{n+1})^2 - 2(\lambda\mu)^{n+1} \right) \\ &= \frac{1}{2} (\lambda^{n+1} - \mu^{n+1})^2 \\ &= \frac{1}{2} (b_{n+1})^2. \end{aligned}$$

When n is even, $n + 1$ is odd and $(\lambda\mu)^{n+1} = -1$. Then we have:

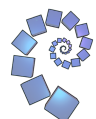
$$\begin{aligned} \sum_{m=0}^{2n} \left(\sum_{r=0}^m a_r \right) &= \frac{1}{2} (\lambda^{2n+2} + \mu^{2n+2} - 2) \\ &= \frac{1}{2} \left((\lambda^{n+1})^2 + (\mu^{n+1})^2 + 2(\lambda\mu)^{n+1} \right) \\ &= \frac{1}{2} (\lambda^{n+1} + \mu^{n+1})^2 \\ &= \frac{1}{2} (a_{n+1})^2. \end{aligned}$$

(iii) From part (i) we have $\left(\sum_{r=0}^n a_r \right)^2 = \left(\frac{1}{\sqrt{2}} b_{n+1} \right)^2 = \frac{1}{2} (b_{n+1})^2$.

We also need:

$$\begin{aligned} \sum_{r=0}^n a_{2r+1} &= a_1 + a_3 + a_5 + \dots + a_{2n+1} \\ &= (\lambda^1 + \lambda^3 + \lambda^5 + \dots + \lambda^{2n+1}) + (\mu^1 + \mu^3 + \mu^5 + \dots + \mu^{2n+1}) \\ &= \lambda \left(1 + \lambda^2 + (\lambda^2)^2 + \dots + (\lambda^2)^n \right) + \mu \left(1 + \mu^2 + (\mu^2)^2 + \dots + (\mu^2)^n \right) \\ &= \frac{\lambda \left((\lambda^2)^{n+1} - 1 \right)}{\lambda^2 - 1} + \frac{\mu \left((\mu^2)^{n+1} - 1 \right)}{\mu^2 - 1} \end{aligned}$$

³What I did, and what I would expect a lot of people to do, is expand $(b_{n+1})^2 = (\lambda^{n+1} - \mu^{n+1})^2$ and try and figure out how it is related to what I have already done. Then I wrote up a solution “going the correct way”.



$\lambda^2 - 1 = 3 + 2\sqrt{2} - 1 = 2(1 + \sqrt{2}) = 2\lambda$, and similarly $\mu^2 - 1 = 2\mu$. We now have:

$$\begin{aligned} \sum_{r=0}^n a_{2r+1} &= \frac{\lambda \left((\lambda^2)^{n+1} - 1 \right)}{\lambda^2 - 1} + \frac{\mu \left((\mu^2)^{n+1} - 1 \right)}{\mu^2 - 1} \\ &= \frac{\lambda \left((\lambda^2)^{n+1} - 1 \right)}{2\lambda} + \frac{\mu \left((\mu^2)^{n+1} - 1 \right)}{2\mu} \\ &= \frac{1}{2} \left((\lambda^{n+1})^2 + (\mu^{n+1})^2 - 2 \right) \end{aligned}$$

Which — as in part (ii) — is equal to $\frac{1}{2} (b_{n+1})^2$ if n is odd and $\frac{1}{2} (a_{n+1})^2$ if n is even.

$$\left(\sum_{r=0}^n a_r \right)^2 - \sum_{r=0}^n a_{2r+1} = \frac{1}{2} (b_{n+1})^2 - \frac{1}{2} (b_{n+1})^2 = 0 \quad \text{if } n \text{ is odd}$$

$$\begin{aligned} \left(\sum_{r=0}^n a_r \right)^2 - \sum_{r=0}^n a_{2r+1} &= \frac{1}{2} (b_{n+1})^2 - \frac{1}{2} (a_{n+1})^2 \\ &= \frac{1}{2} \left((\lambda^{n+1})^2 + (\mu^{n+1})^2 - 2(\lambda\mu)^{n+1} \right) \\ &\quad - \frac{1}{2} \left((\lambda^{n+1})^2 + (\mu^{n+1})^2 + 2(\lambda\mu)^{n+1} \right) \\ &= -2(\lambda\mu)^{n+1} \\ &= -2 \times (-1)^{n+1} = 2 \quad \text{if } n \text{ is even.} \end{aligned}$$

