1 The first 10 cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

(i) Substituting $y = k - x$ gives:

\[
x^3 + (k - x)^3 = k z^3
\]
\[
x^3 + k^3 - 3k^2 x + 3kx^2 - x^3 = k z^3
\]
\[
k^2 - 3kx + 3x^2 = z^3 \quad \text{since } k > 0.
\]

Note that we need to state something like $k \neq 0$ before we divide by $k$.

We then have:

\[
\frac{4z^3 - k^2}{3} = \frac{4(k^2 - 3kx + 3x^2) - k^2}{3}
\]
\[
= \frac{3k^2 - 12kx + 12x^2}{3}
\]
\[
= k^2 - 4kx + 4x^2
\]
\[
= (k - 2x)^2.
\]

You can if you like use $k = y + x$ to re-write the last line as $(y - x)^2$.

A perfect square is greater than or equal to 0, so we have $4z^3 - k^2 \geq 0 \implies z^3 \geq \frac{1}{4}k^2$.

For the other part of the inequality use:

\[
z^3 = k^2 - 3kx + 3x^2
\]
\[
= k^2 - 3x(k - x)
\]
\[
= k^2 - 3xy
\]

and as $x, y > 0$ we have $z^3 < k^2$. Therefore we have $\frac{1}{4}k^2 \leq z^3 < k^2$.

When $k = 20$ we have $100 \leq z^3 < 400$, so $z$ must be 5, 6 or 7 (this is why you were asked to work out the first few cubes!). Testing each of these in $\frac{4z^3 - k^2}{3}$ shows that only $z = 7$ results in a perfect square. Using $\frac{4z^3 - k^2}{3} = 18^2 = (y - x)^2$ and $x + y = k = 20$, and assuming WLOG\(^1\) that $y \geq x$, we can solve the simultaneous equations to get $x = 1, y = 19$. You can then check that $1^3 + 19^3 = 20 \times 7^3$ if you wish.

---

\(^1\)Without Loss Of Generality
(ii) Follow the same method as part (i)!

Substituting \( y = z^2 - x \) gives:

\[
\begin{align*}
    x^3 + (z^2 - x)^3 &= kz^3 \\
    x^6 + z^6 - 3z^4x + 3z^2x^2 - x^3 &= kz^3 \\
    z^4 - 3z^2x + 3x^2 &= kz & \text{ since } z > 0.
\end{align*}
\]

We are then looking for something that will give us a perfect square. Comparison with part (i) leads us to:

\[
\begin{align*}
    \frac{4kz - z^4}{3} &= \frac{4(z^4 - 3z^2x + 3x^2) - z^4}{3} \\
    &= z^4 - 4z^2x + 4x^2 \\
    &= (z^2 - 2x)^2 \\
    &= (y - x)^2.
\end{align*}
\]

We then have \( 4kz - z^4 \geq 0 \implies z^3 \leq 4k \) (since \( z > 0 \) we can divide by \( z \) without changing the inequality direction).

We also have:

\[
\begin{align*}
    kz &= z^4 - 3z^2x + 3x^2 \\
    &= z^4 - 3x(z^2 - x) \\
    &= z^4 - 3xy
\end{align*}
\]

and as \( x, y > 0 \) we have \( kz < z^4 \implies k < z^3 \). Hence we have \( k < z^3 \leq 4k \). With \( k = 19 \) this gives \( 19 < z^3 \leq 76 \) and so \( z = 3 \) or \( 4 \). Both of these give perfect squares for \( \frac{4kz - z^4}{3} = (y - x)^2 \).

\( z = 3 \) gives \( x + y = z^2 = 9 \) and \( (y - x)^2 = 49 \), so \( y = 8 \) and \( x = 1 \).

\( z = 4 \) gives \( x + y = z^2 = 16 \) and \( (y - x)^2 = 16 \), so \( y = 10 \) and \( x = 6 \).

Again, you can check that these solve \( x^3 + y^3 = kz^3 \).
It is helpful to define a coordinate system for the tetrahedron. Let the line \( AB \) be on the \( x \)-axis so that the midpoint of \( AB \) is at the origin. This means that we have \( A = (-\frac{1}{2}, 0, 0) \) and \( B = (\frac{1}{2}, 0, 0) \). Using Pythagoras' theorem in \( \triangle AOC \) gives \( C = (0, \frac{\sqrt{3}}{2}, 0) \).

(i) The centroid of \( \triangle ABC \) is a third of the distance from the centre of \( AB \) to \( C \) which gives \( P = (0, \frac{\sqrt{3}}{6}, 0) \). You can then use Pythagoras' theorem to find lengths \( PA \) and \( PD \):

\[
PA^2 = \frac{1}{4} + \frac{3}{36} = \frac{1}{3}
\]

\[
PD^2 = 1 - \frac{1}{3} = \frac{2}{3}
\]

and hence \( PD = \sqrt{\frac{2}{3}} \).

(ii) The angle between two adjacent faces is given by, e.g., \( \angle DOC = \angle DOP \). Using the right-angled triangle \( \triangle DOP \) gives \( \cos(\angle DOP) = \frac{\frac{1}{6}}{\frac{\sqrt{3}}{2}} = \frac{1}{3} \).

(iii) The centre of the sphere, \( S \), must lie on the line \( DP \) by symmetry. Let \( X \) be where the sphere meets the face \( ABD \).

We know that \( OP = OX = \frac{\sqrt{3}}{6} \), \( OD = \frac{\sqrt{3}}{2} \) and \( DP = \frac{\sqrt{6}}{3} \). Using the right-angled triangle \( \triangle DXS \) we have:

\[
DS^2 = XS^2 + XD^2
\]

\[
\left( \frac{\sqrt{6}}{3} - r \right)^2 = r^2 + \left( \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6} \right)^2
\]

\[
\frac{6}{9} - \frac{2\sqrt{6}}{3} r + r^2 = r^2 + \frac{1}{3}
\]

\[
\frac{2\sqrt{6}}{3} r = \frac{1}{3}
\]

\[
r = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12}
\]
The curve $xy = v$ is symmetric in $y = x$ (which we can see as if we interchange $x$ and $y$ we get the same curve) and $y = -x$ (which we can see by using $x' = -y$ and $y' = -x$).

The curve $x^4 + y^4 = u$ is symmetric in $y = x$, $y = -x$, the $x$-axis (which can be seen by substituting $y' = -y$) and the $y$-axis (substitute $x' = -x$).

If $A = (\alpha, \beta)$ then $B = (\beta, \alpha)$, $C = (-\alpha, -\beta)$ and $D = (-\beta, -\alpha)$.

To show that $ABCD$ is a quadrilateral, you can show that $AB$ is perpendicular to $BC$. We have:

$$m_{AB} \times m_{BC} = \frac{\beta - \alpha}{\alpha - \beta} \times \frac{\alpha + \beta}{\beta + \alpha} = -1.$$ 

Therefore $AB$ and $BC$ are perpendicular. You can then use the same technique to show that all of the four sides are perpendicular, or you can use symmetry in $y = x$ and $y = -x$ to show this.

To find the area:

$$AB^2 = (\alpha - \beta)^2 + (\beta - \alpha)^2 = 2(\alpha - \beta)^2$$

$$BC^2 = (\beta + \alpha)^2 + (\alpha + \beta)^2 = 2(\alpha + \beta)^2.$$ 

Remembering that $\alpha > \beta$ the area is:

$$AB \times BC = \sqrt{2}(\alpha - \beta) \times \sqrt{2}(\alpha + \beta) = 2(\alpha^2 - \beta^2).$$

The question asks us for the area in terms of $u$ and $v$. Since $\alpha$ and $\beta$ satisfy the equations of the curves we have $\alpha^4 + \beta^4 = u$ and $\alpha\beta = v$.

Considering $(\alpha^2 - \beta^2)^2$ gives us:

$$(\alpha^2 - \beta^2)^2 = \alpha^4 - 2\alpha^2\beta^2 + \beta^4$$

$$= \alpha^4 + \beta^4 - 2(\alpha\beta)^2$$

$$= u - 2v^2$$

So the area is $2\sqrt{u - 2v^2}$.

Then for $u = 81$ and $v = 4$ the area is $2\sqrt{81 - 32} = 2\sqrt{49} = 14$. 

STEP 2 Miscellaneous: Solutions 4
We can deduce that
\[ p(x) - 1 = (x - 1)^5 \times q(x) \]  
(*)
where \( q(x) \) is a quartic.

(i) Using (*) we have \( p(1) - 1 = (1 - 1)^5 \times q(1) \), so \( p(1) = 1 \).

(ii) Differentiating (*) gives:
\[
p'(x) = 5(x - 1)^4 \times q(x) + (x - 1)^5 \times q'(x) \\
= (x - 1)^4(5q(x) + (x - 1)q'(x))
\]
and therefore \( p'(x) \) is divisible by \( (x - 1)^4 \).

(iii) In a similar way to before we can write \( p(x) + 1 = (x + 1)^5 \times q_2(x) \). Substituting in \( x = -1 \) gives \( p(-1) = -1 \) and differentiating can be used to show that \( p'(x) \) is divisible by \( (x + 1)^4 \).

We can now write \( p'(x) = k(x - 1)^4(x + 1)^4 = k(x^2 - 1)^4 = k(x^8 - 4x^6 + 6x^4 - 4x^2 + 1) \).

Integrating gives:
\[
p(x) = k\left(\frac{1}{5}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right) + c
\]
and using \( p(1) = 1 \) and \( p(-1) = -1 \) we have:
\[
k\left(\frac{1}{5} - \frac{4}{7} + \frac{6}{5} - \frac{4}{3} + 1\right) + c = 1 \\
k\left(-\frac{1}{9} + \frac{4}{7} - \frac{6}{5} + \frac{4}{3} + 1\right) + c = -1
\]

Adding these two together gives \( c = 0 \) and then the first one gives us:
\[
k\left(\frac{35}{315} - \frac{180}{315} + \frac{378}{315} - \frac{420}{315} + \frac{315}{315}\right) = 1 \implies k = \frac{315}{128}.
\]

So \( p(x) = \frac{315}{128}\left(\frac{1}{5}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right) \).
\[ F_3 = 2, \ F_4 = 3, \ F_5 = 5, \ F_6 = 8, \ F_7 = 13, \ F_8 = 21, \ F_9 = 34 \text{ and } F_{10} = 55. \]

(i) We have \( F_i = F_{i-1} + F_{i-2} \) and, as long as \( i \geq 4 \), we have \( F_{i-2} < F_{i-1} \) (when \( i = 3 \) we have equality as \( F_1 = F_2 \)). Hence we have \( F_i < 2F_{i-1} \) and so \( \frac{1}{F_i} > \frac{1}{2F_{i-1}} \). We also have \( \frac{1}{F_{i-1}} > \frac{1}{2F_{i-2}} \) (if \( i \geq 5 \)) and so \( \frac{1}{F_i} > \frac{1}{4F_{i-2}} \) etc.

We now have:

\[
S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \cdots \\
S > \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{2F_4} + \frac{1}{2F_5} + \frac{1}{2F_6} + \cdots \\
S > \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{2F_3} + \frac{1}{4F_4} + \frac{1}{8F_5} + \cdots \\
S > \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right)
\]

Using the values of \( F_1, F_2, F_3 \) and the sum of an infinite GP gives us:

\[ S > 1 + 1 + \frac{1}{2} \times 2 = 3. \]

In a similar way to above we have \( F_i > 2F_{i-2} \) (for \( i \geq 4 \)) and so \( \frac{1}{F_i} < \frac{1}{2F_{i-2}} \). This is slightly different to before as you have to split up odd and even terms.

\[
S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \cdots \\
S = \left( \frac{1}{F_1} + \frac{1}{F_2} \right) + \left( \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \cdots \right) + \left( \frac{1}{F_3} + \frac{1}{F_5} + \frac{1}{F_7} + \cdots \right) \\
S < \left( \frac{1}{F_1} + \frac{1}{F_2} \right) + \left( \frac{1}{F_4} + \frac{1}{2F_4} + \frac{1}{4F_4} + \cdots \right) + \left( \frac{1}{F_3} + \frac{1}{2F_3} + \frac{1}{4F_3} + \cdots \right) \\
S < 1 + 1 + \frac{1}{3} \times 2 + \frac{1}{2} \times 2 = \frac{32}{3}
\]

(ii) For this part, use the same approach but take more terms before using the geometric series.

\[
S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \cdots \\
S > \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} \left( 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) \\
S > 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{5}{2} \times 2 = 1 + 1 + \frac{37}{30}
\]

Hence \( S > \frac{37}{30} > \frac{36}{30} = 3.2 \)
And for the upper limit:

\[ S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \ldots \]

\[ S = \left( \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} \right) + \left( \frac{1}{F_5} + \frac{1}{F_7} + \ldots \right) + \left( \frac{1}{F_6} + \frac{1}{F_8} + \ldots \right) \]

\[ S < \left( \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} \right) + \frac{1}{F_5} \left( 1 + \frac{1}{2} + \ldots \right) + \frac{1}{F_6} \left( 1 + \frac{1}{2} + \ldots \right) \]

\[ S < 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 + \frac{1}{8} \times 2 \]

and so \( S < 3\frac{29}{60} < 3\frac{1}{2} \).
6 (i) Setting \(a = 1, \ b = -1, \ x_n = x_{n+1} = x\) and \(y_n = y_{n+1} = y\) gives the simultaneous equations:
\[
\begin{align*}
x &= x^2 - y^2 + 1 \\
y &= 2xy + 1
\end{align*}
\]
Using the second equation we have \(x = \frac{y - 1}{2y}\), which we can substitute into the first equation to get:
\[
\begin{align*}
y - 1 &= \left(\frac{y - 1}{2y}\right)^2 - y^2 + 1 \\
2y(y - 1) &= (y - 1)^2 - 4y^2 \times y^2 + 4y^2 \\
2y^2 - 2y &= y^2 - 2y + 1 - 4y^4 + 4y^2 \\
4y^4 - 3y^2 - 1 &= 0 \\
(y - 1)(4y^3 + 4y^2 + y + 1) &= 0 \\
(y - 1)(y + 1)(4y^2 + 1) &= 0
\end{align*}
\]
Therefore \(y = 1\) or \(y = -1^2\). Using \(x = \frac{y - 1}{2y}\) gives the values as \((x_1, y_1) = (0, 1)\) and \((1, -1)\).

(ii) Taking \((x_1, y_1) = (-1, 1)\) gives \((x_2, y_2) = (a, b)\) and \((x_3, y_3) = (a^2 - b^2 + a, 2ab + b + 2)\). If the sequence is to have period 2 then we need \((x_1, y_1) = (x_3, y_3) \neq (x_2, y_2)\).

Using \((x_1, y_1) = (x_3, y_3)\) we have:
\[
\begin{align*}
a^2 - b^2 + a &= -1 \\
2ab + b + 2 &= 1
\end{align*}
\]
Similarly to before, we have \(a = \frac{-b - 1}{2b}\) and so:
\[
\begin{align*}
\left(\frac{-b - 1}{2b}\right)^2 - b^2 - \frac{b + 1}{2b} &= -1 \\
(-b - 1)^2 - 4b^4 - 2b(b + 1) &= -4b^2 \\
b^2 + 2b + 1 - 4b^4 - 2b^2 - 2b &= -4b^2 \\
4b^4 - 3b^2 - 1 &= 0
\end{align*}
\]
This last equation is identical to the one in \(y\) for part (i), so we have \(b = \pm 1\). This gives \((a, b) = (0, -1)\) or \((-1, 1)\), but since the second gives \((x_2, y_2) = (x_1, y_1)\) (i.e. sequence is constant, not period 2) we discard that one to leave us with just one solution, \((a, b) = (0, -1)\).

There is a neat solution here, where you can spot that if you let \(a = -x\) and \(b = -y\) you get the same equations as in part (i), which means you can deduce the values of \(a\) and \(b\) without solving the simultaneous equations.

---

2The equation \(4y^2 = -1\) has no real solutions, and since we want \((x, y)\) to be a point in the Cartesian plane, we want \(y\) to be real.
(i) The binomial expansion gives:

\[
\left(1 + \frac{k}{100}\right)^{\frac{1}{2}} = 1 + \frac{1}{2} \times \left(\frac{k}{100}\right) + \frac{1}{2!} \times \frac{1}{2} \times -\frac{1}{2} \times \left(\frac{k}{100}\right)^{2} + \frac{1}{3!} \times \frac{1}{2} \times -\frac{1}{2} \times -\frac{3}{2} \times \left(\frac{k}{100}\right)^{3} + \ldots
\]

\[\approx 1 + \frac{k}{200} - \frac{k^{2}}{80000} + \frac{k^{3}}{16000000} \]

(a) Substituting \(k = 8\) gives:

\[
\left(1 + \frac{8}{100}\right)^{\frac{1}{2}} = \left(\frac{108}{100}\right)^{\frac{1}{2}} = \left(\frac{3 \times 36}{100}\right)^{\frac{1}{2}} = \frac{6}{10} \times \sqrt{3}
\]

Using the binomial expansion (with \(k = 8\)) gives:

\[
\frac{6}{10} \times \sqrt{3} \approx 1 + \frac{8}{200} - \frac{8 \times 8}{80000} + \frac{8 \times 8 \times 8}{16000000} = 1 + \frac{4}{100} - \frac{8}{10000} + \frac{8}{1000000} = 1.040032 - 0.0008 = 1.039232
\]

Then the approximation for \(\sqrt{3}\) is given by:

\[
\sqrt{3} \approx \frac{1.039232}{0.6} = \frac{10.39232}{6}
\]

Carrying out the division gives \(\sqrt{3} \approx 1.73205\).

(b) Here we need to find a suitable value of \(k\), and remember that we want \(k\) to be small in order to get a good approximation. Comparing to the previous part, we want \(100 + k = a^{2} \times 6\). Starting with \(a = 3\) we have:

\[
3^{2} \times 6 = 54 \implies k = -46
\]
\[
4^{2} \times 6 = 96 \implies k = -4
\]
\[
5^{2} \times 6 = 150 \implies k = 50
\]
$k = 50$ is not a great choice as it is not small, so take $k = -4$.

Substituting $k = -4$ gives:

$$
\left( \frac{96}{100} \right)^{\frac{1}{2}} = \frac{4}{10} \times \sqrt{6}
\approx 1 - \frac{4}{200} - \frac{4 \times 4}{80000} - \frac{4 \times 4 \times 4}{16000000}
= 1 - \frac{2}{100} - \frac{2}{10000} - \frac{4}{1000000}
= 1 - 0.020204
= 0.979796
$$

And so we have $\sqrt{6} \approx 0.979796 \div 4$, i.e. $\sqrt{6} \approx 2.44949$.

(ii) The first two terms of the binomial expansion of $\left(1 + \frac{k}{1000}\right)^{\frac{1}{3}}$ gives us:

$$
\left(1 + \frac{k}{1000}\right)^{\frac{1}{3}} \approx 1 + \frac{k}{3000}.
$$

By comparing to the previous part, we want to find a value of $k$ so that $1000 + k = a^3 \times 3$, where $k$ is small compared to 1000. The value of $a$ which gives the smallest $k$ is $a = 7$, which gives $1000 + k = 1029 \implies k = 29$. We then have:

$$
\left( \frac{1029}{1000} \right)^{\frac{1}{3}} \approx 1 + \frac{29}{3000}
\frac{7}{10} \times \sqrt[3]{3} \approx \frac{3029}{3000}
\sqrt[3]{3} \approx \frac{3029}{3000} \times \frac{10}{7} \quad \text{and so}
\sqrt[3]{3} \approx \frac{3029}{2100} \quad \text{as required.}$$
You should start by drawing a large and clear diagram, maybe something like below:

Then we have $CX = b - r$ and $BX = c - r$ (adjacent sides of a kite). Hence $a = (b - r) + (c - r)$ and so $2r = b + c - a$.

We now have another diagram with the circumcircle shown as well as the incircle.

Since $\triangle ABC$ is right-angled, $BC$ is a diameter of $S_2$. Hence the radius of $S_2$ satisfies $2r_2 = a$.

The area of $S_2$ is $\pi r_2^2$ and the area between $S_1$ and the triangle is $\frac{1}{2}bc - \pi r^2$. Using the given fact about the ratio of these we have:

$$R\pi r_2^2 = \frac{1}{2}bc - \pi r^2$$
The result we are trying to obtain does not contain \( r \) or \( r^2 \), so we substitute for these.

\[
R\pi \left( \frac{a}{2} \right)^2 = \frac{1}{2} bc - \pi \left( \frac{b + c - a}{2} \right)^2 \\
R\pi = \frac{2bc}{a^2} - \pi \left( \frac{b + c - a}{a} \right)^2 \\
R\pi = \frac{2bc}{a^2} - \pi (q - 1)^2
\]

This is starting to look promising, but the \( \frac{2bc}{a^2} \) needs writing in terms of \( q \). We have:

\[
q^2 = \left( \frac{b + c}{a} \right)^2 \\
= \frac{b^2 + c^2 + 2bc}{a^2} \\
= \frac{a^2 + 2bc}{a^2} \quad \text{using Pythagoras’ theorem} \\
= 1 + \frac{2bc}{a^2}
\]

So we now have:

\[
R\pi = q^2 - 1 - \pi (q - 1)^2 \\
= q^2 - 1 - \pi(q^2 - 2q + 1) \\
= q^2 - 1 - \pi q^2 + 2\pi q - \pi \\
= -(\pi - 1)q^2 + 2\pi q - (\pi + 1) \quad \text{as required}
\]

Note that this is a quadratic in \( q \), and will have a maximum when \( \frac{d}{dq} (\pi R) = 0 \), i.e. when

\[
q = \frac{2\pi}{2(\pi - 1)} = \frac{\pi}{\pi - 1}. \quad \text{This gives}
\]

\[
\pi R_{\max} = -(\pi - 1) \left( \frac{\pi}{\pi - 1} \right)^2 + 2\pi \left( \frac{\pi}{\pi - 1} \right) - (\pi + 1) \\
= -\frac{\pi^2}{\pi - 1} + \frac{2\pi^2}{\pi - 1} - (\pi + 1) \\
= \frac{\pi^2}{\pi - 1} - \frac{(\pi + 1)(\pi - 1)}{\pi - 1} \\
= \frac{\pi^2 - \pi^2 + 1}{\pi - 1} \\
= \frac{1}{\pi - 1}
\]

And so since \( R_{\max} = \frac{1}{\pi(\pi - 1)} \) we have \( R \leq \frac{1}{\pi(\pi - 1)} \).
One stumbling block is not reading all the information in the “stem”! Since you are told that $\lambda = 1 + \sqrt{2}$ you know that $\lambda - 1 = \sqrt{2}$ etc.

We have:

$$\sum_{r=0}^{n} b_r = (\lambda^0 - \mu^0) + (\lambda^1 - \mu^1) + (\lambda^2 - \mu^2) + \ldots + (\lambda^n - \mu^n)$$

$$= (1 + \lambda^1 + \lambda^2 + \ldots + \lambda^n) - (1 + \mu^1 + \mu^2 + \ldots + \mu^n)$$

$$= \frac{\lambda^{n+1} - 1}{\lambda - 1} - \frac{\mu^{n+1} - 1}{\mu - 1}$$

$$= \frac{\lambda^{n+1} - 1}{\sqrt{2}} - \frac{\mu^{n+1} - 1}{-\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} (\lambda^{n+1} - \mu^{n+1}) - 2 \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} a_{n+1} - \sqrt{2}$$

Similarly:

$$\sum_{r=0}^{n} a_r = (\lambda^0 + \mu^0) + (\lambda^1 + \mu^1) + (\lambda^2 + \mu^2) + \ldots + (\lambda^n + \mu^n)$$

$$= (1 + \lambda^1 + \lambda^2 + \ldots + \lambda^n) + (1 + \mu^1 + \mu^2 + \ldots + \mu^n)$$

$$= \frac{\lambda^{n+1} - 1}{\lambda - 1} + \frac{\mu^{n+1} - 1}{\mu - 1}$$

$$= \frac{\lambda^{n+1} - 1}{\sqrt{2}} + \frac{\mu^{n+1} - 1}{-\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} (\lambda^{n+1} - \mu^{n+1})$$

$$= \frac{1}{\sqrt{2}} b_{n+1}$$

(ii) Here we have a “nested sum”. Start by evaluating the “inner sum”.

$$\sum_{m=0}^{2n} \left( \sum_{r=0}^{m} a_r \right) = \sum_{m=0}^{2n} \left( \frac{1}{\sqrt{2}} b_{m+1} \right)$$

$$= \sum_{m=0}^{2n+1} \left( \frac{1}{\sqrt{2}} b_{m} \right) \quad \text{since } b_0 = 0$$

$$= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} a_{(2n+1)+1} - \sqrt{2} \right)$$

$$= \frac{1}{2} (a_{2n+2} - 2)$$

$$= \frac{1}{2} (\lambda^{2n+2} + \mu^{2n+2} - 2)$$
We are trying to get something like \((b_{n+1})^2\). Noting that \(\lambda \times \mu = 1 - 2 = -1\), and using the fact that \(n\) is odd, so that \(n + 1\) is even and \((\lambda\mu)^{n+1} = 1\) we have:

\[
\sum_{m=0}^{2n} \left( \sum_{r=0}^{m} a_r \right) = \frac{1}{2} \left( \lambda^{2n+2} + \mu^{2n+2} - 2 \right)
\]

\[
= \frac{1}{2} \left( (\lambda^{n+1})^2 + (\mu^{n+1})^2 - 2 (\lambda\mu)^{n+1} \right)
\]

\[
= \frac{1}{2} (\lambda^{n+1} - \mu^{n+1})^2
\]

\[
= \frac{1}{2} (b_{n+1})^2.
\]

When \(n\) is even, \(n + 1\) is odd and \((\lambda\mu)^{n+1} = -1\). Then we have:

\[
\sum_{m=0}^{2n} \left( \sum_{r=0}^{m} a_r \right) = \frac{1}{2} \left( \lambda^{2n+2} + \mu^{2n+2} - 2 \right)
\]

\[
= \frac{1}{2} \left( (\lambda^{n+1})^2 + (\mu^{n+1})^2 + 2 (\lambda\mu)^{n+1} \right)
\]

\[
= \frac{1}{2} (\lambda^{n+1} + \mu^{n+1})^2
\]

\[
= \frac{1}{2} (a_{n+1})^2.
\]

(iii) From part (i) we have \(\left( \sum_{r=0}^{n} a_r \right)^2 = \left( \frac{1}{\sqrt{2}} b_{n+1} \right)^2 = \frac{1}{2} (b_{n+1})^2\).

We also need:

\[
\sum_{r=0}^{n} a_{2r+1} = a_1 + a_3 + a_5 + \ldots + a_{2n+1}
\]

\[
= (\lambda + \lambda^3 + \lambda^5 + \ldots + \lambda^{2n+1}) + (\mu + \mu^3 + \mu^5 + \ldots + \mu^{2n+1})
\]

\[
= \lambda \left( 1 + \lambda^2 + (\lambda^2)^2 + \ldots + (\lambda^2)^n \right) + \mu \left( 1 + \mu^2 + (\mu^2)^2 + \ldots + (\mu^2)^n \right)
\]

\[
= \frac{\lambda \left( (\lambda^2)^{n+1} - 1 \right)}{1 + \mu (\mu^2)^{n+1} - \mu^2 - 1}
\]

\[
= \frac{\lambda \left( (\lambda^2)^{n+1} - 1 \right)}{\mu^2 - 1}
\]

\[
\text{What I did, and what I would expect a lot of people to do, is expand } (b_{n+1})^2 = (\lambda^{n+1} - \mu^{n+1})^2 \text{ and try and figure out how it is related to what I have already done. Then I wrote up a solution "going the correct way".}
\[
\lambda^2 - 1 = 3 + 2\sqrt{2} - 1 = 2(1 + \sqrt{2}) = 2\lambda, \text{ and similarly } \mu^2 - 1 = 2\mu. \text{ We now have:}
\]
\[
\sum_{r=0}^{n} a_{2r+1} = \frac{\lambda \left( (\lambda^2)^{n+1} - 1 \right)}{\lambda^2 - 1} + \frac{\mu \left( (\mu^2)^{n+1} - 1 \right)}{\mu^2 - 1}
\]
\[
= \frac{\lambda \left( (\lambda^2)^{n+1} - 1 \right)}{2\lambda} + \frac{\mu \left( (\mu^2)^{n+1} - 1 \right)}{2\mu}
\]
\[
= \frac{1}{2} \left( (\lambda^{n+1})^2 + (\mu^{n+1})^2 - 2 \right)
\]

Which — as in part (ii) — is equal to \(\frac{1}{2} (b_{n+1})^2\) if \(n\) is odd and \(\frac{1}{2} (a_{n+1})^2\) if \(n\) is even.

\[
\left( \sum_{r=0}^{n} a_r \right)^2 - \sum_{r=0}^{n} a_{2r+1} = \frac{1}{2} (b_{n+1})^2 - \frac{1}{2} (b_{n+1})^2 = 0 \quad \text{if } n \text{ is odd}
\]
\[
\left( \sum_{r=0}^{n} a_r \right)^2 - \sum_{r=0}^{n} a_{2r+1} = \frac{1}{2} (b_{n+1})^2 - \frac{1}{2} (a_{n+1})^2
\]
\[
= \frac{1}{2} \left( (\lambda^{n+1})^2 + (\mu^{n+1})^2 - 2 (\lambda\mu)^{n+1} \right) - \frac{1}{2} \left( (\lambda^{n+1})^2 + (\mu^{n+1})^2 + 2 (\lambda\mu)^{n+1} \right)
\]
\[
= -2 (\lambda\mu)^{n+1}
\]
\[
= -2 \times (-1)^{n+1} = 2 \quad \text{if } n \text{ is even.}