# STEP Support Programme

## STEP 2 Miscellaneous Questions: Solutions

**1** The first 10 cubes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000.

**(i)** Substituting $y = k - x$ gives:

$$\begin{aligned} x^3 + (k-x)^3 &= kz^3 \\ \cancel{x^3} + k^3 - 3k^2x + 3kx^2 - \cancel{x^3} &= kz^3 \\ k^2 - 3kx + 3x^2 &= z^3 \qquad \text{since } k > 0. \end{aligned}$$

Note that we need to state something like $k \neq 0$ before we divide by $k$.

We then have:

$$\begin{aligned} \frac{4z^3 - k^2}{3} &= \frac{4(k^2 - 3kx + 3x^2) - k^2}{3} \\ &= \frac{3k^2 - 12kx + 12x^2}{3} \\ &= k^2 - 4kx + 4x^2 \\ &= (k-2x)^2 \,. \end{aligned}$$

You can if you like use $k = y + x$ to re-write the last line as $(y-x)^2$.

A perfect square is greater than or equal to 0, so we have $4z^3 - k^2 \geqslant 0 \implies z^3 \geqslant \frac{1}{4}k^2$.

For the other part of the inequality use:

$$\begin{aligned} z^3 &= k^2 - 3kx + 3x^2 \\ &= k^2 - 3x(k-x) \\ &= k^2 - 3xy \end{aligned}$$

and as $x, y > 0$ we have $z^3 < k^2$. Therefore we have $\frac{1}{4}k^2 \leqslant z^3 < k^2$.

When $k = 20$ we have $100 \leqslant z^3 < 400$, so $z$ must be 5, 6 or 7 (this is why you were asked to work out the first few cubes!). Testing each of these in $\dfrac{4z^3 - k^2}{3}$ shows that only $z = 7$ results in a perfect square. We have $\dfrac{4z^3 - k^2}{3} = 18^2 = (y-x)^2$ and since we are given that $x < y$ this gives $y - x = 18$. We also have $x + y = k = 20$, and we can solve these simultaneous equations to get $x = 1$, $y = 19$. You can then check that $1^3 + 19^3 = 20 \times 7^3$ if you wish.

**(ii)** Follow the same method as part **(i)**!

Substituting $y = z^2 - x$ gives:

$$\begin{aligned} x^3 + (z^2 - x)^3 &= kz^3 \\ \cancel{x^3} + z^6 - 3z^4x + 3z^2x^2 - \cancel{x^3} &= kz^3 \\ z^4 - 3z^2x + 3x^2 &= kz \qquad \text{since } z > 0. \end{aligned}$$

We are then looking for something that will give us a perfect square. Comparison with part **(i)** leads us to:

$$\begin{aligned} \frac{4kz - z^4}{3} &= \frac{4(z^4 - 3z^2x + 3x^2) - z^4}{3} \\ &= z^4 - 4z^2x + 4x^2 \\ &= (z^2 - 2x)^2 \\ &= (y - x)^2 \,. \end{aligned}$$

We then have $4kz - z^4 \geqslant 0 \implies z^3 \leqslant 4k$ (since $z > 0$ we can divide by $z$ without changing the inequality direction).

We also have:

$$\begin{aligned} kz &= z^4 - 3z^2x + 3x^2 \\ &= z^4 - 3x(z^2 - x) \\ &= z^4 - 3xy \end{aligned}$$

and as $x, y > 0$ we have $kz < z^4 \implies k < z^3$. Hence we have $k < z^3 \leqslant 4k$. With $k = 19$ this gives $19 < z^3 \leqslant 76$ and so $z = 3$ or $4$. Both of these give perfect squares for $\dfrac{4kz - z^4}{3} = (y - x)^2$.

$z = 3$ gives $x + y = z^2 = 9$ and $(y - x)^2 = 49$, so $y = 8$ and $x = 1$.

$z = 4$ gives $x + y = z^2 = 16$ and $(y - x)^2 = 16$, so $y = 10$ and $x = 6$.

Again, you can check that these solve $x^3 + y^3 = kz^3$.

**2** It is helpful to define a coordinate system for the tetrahedron. Let the line $AB$ be on the $x$-axis so that the midpoint of $AB$ is at the origin. This means that we have $A = \left(-\frac{1}{2}, 0, 0\right)$ and $B = \left(\frac{1}{2}, 0, 0\right)$. Using Pythagoras' theorem in $\triangle AOC$ gives $C = \left(0, \frac{\sqrt{3}}{2}, 0\right)$.

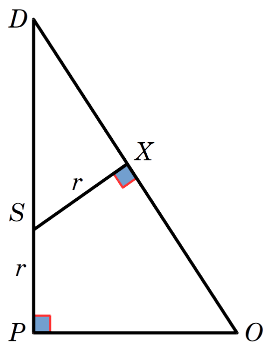
**(i)** The *centroid* of $\triangle ABC$ is a third of the distance from the centre of $AB$ to $C$ which gives $P = \left(0, \frac{\sqrt{3}}{6}, 0\right)$. You can the use Pythagoras' theorem to find lengths $PA$ and $PD$:

$$PA^2 = \frac{1}{4} + \frac{3}{36} = \frac{1}{3}$$
$$PD^2 = 1 - \frac{1}{3} = \frac{2}{3}$$

and hence $PD = \sqrt{\frac{2}{3}}$.

**(ii)** The angle between two adjacent faces is given by, e.g., $\angle DOC = \angle DOP$. Using the right-angled triangle $\triangle DOP$ gives $\cos(\angle DOP) = \dfrac{\frac{1}{6}\sqrt{3}}{\frac{1}{2}\sqrt{3}} = \frac{1}{3}$.

**(iii)** The centre of the sphere, $S$, must lie on the line $DP$ by symmetry. Let $X$ be where the sphere meets the face $ABD$.

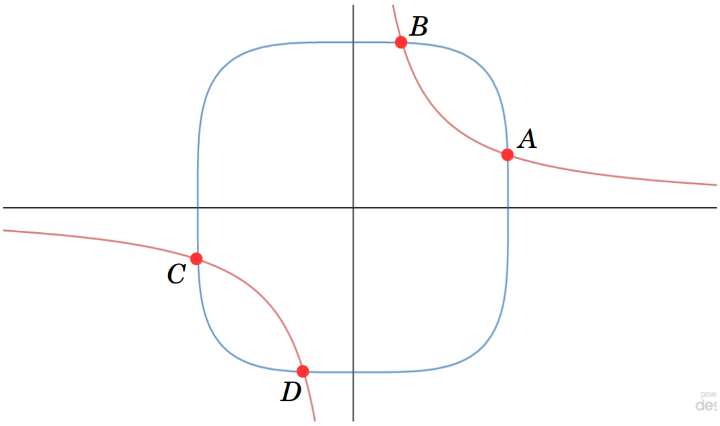



We know that $OP = OX = \frac{\sqrt{3}}{6}$, $OD = \frac{\sqrt{3}}{2}$ and $DP = \frac{\sqrt{6}}{3}$. Using the right-angled triangle $\triangle DXS$ we have:

$$DS^2 = XS^2 + XD^2$$
$$\left(\frac{\sqrt{6}}{3} - r\right)^2 = r^2 + \left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{6}\right)^2$$
$$\frac{6}{9} - \frac{2\sqrt{6}}{3}r + \cancel{r^2} = \cancel{r^2} + \frac{1}{3}$$
$$\frac{2\sqrt{6}}{3}r = \frac{1}{3}$$
$$r = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12}$$

**3** The curve $xy = v$ is symmetric in $y = x$ (which we can see as if we interchange $x$ and $y$ we get the same curve) and $y = -x$ (which we can see by using $x' = -y$ and $y' = -x$).

The curve $x^4 + y^4 = u$ is symmetric in $y = x$, $y = -x$, the $x$-axis (which can be seen by substituting $y' = -y$) and the $y$-axis (substitute $x' = -x$).




If $A = (\alpha, \beta)$ then $B = (\beta, \alpha)$, $C = (-\alpha, -\beta)$ and $D = (-\beta, -\alpha)$.

To show that $ABCD$ is a quadrilateral, you can show that $AB$ is perpendicular to $BC$. We have:

$$m_{AB} \times m_{BC} = \frac{\beta - \alpha}{\alpha - \beta} \times \frac{\alpha + \beta}{\beta + \alpha} = -1\,.$$

Therefore $AB$ and $BC$ are perpendicular. You can then use the same technique to show that all of the four sides are perpendicular, or you can use symmetry in $y = x$ and $y = -x$ to show this.

To find the area:

$$AB^2 = (\alpha - \beta)^2 + (\beta - \alpha)^2 = 2(\alpha - \beta)^2$$
$$BC^2 = (\beta + \alpha)^2 + (\alpha + \beta)^2 = 2(\alpha + \beta)^2\,.$$

Remembering that $\alpha > \beta$ the area is:

$$AB \times BC = \sqrt{2}(\alpha - \beta) \times \sqrt{2}(\alpha + \beta) = 2(\alpha^2 - \beta^2)\,.$$

The question asks us for the area in terms of $u$ and $v$. Since $\alpha$ and $\beta$ satisfy the equations of the curves we have $\alpha^4 + \beta^4 = u$ and $\alpha\beta = v$.

Considering $(\alpha^2 - \beta^2)^2$ gives us:

$$\begin{aligned}(\alpha^2 - \beta^2)^2 &= \alpha^4 - 2\alpha^2\beta^2 + \beta^4 \\ &= \alpha^4 + \beta^4 - 2\,(\alpha\beta)^2 \\ &= u - 2v^2\end{aligned}$$

So the area is $2\sqrt{u - 2v^2}$.

Then for $u = 81$ and $v = 4$ the area is $2\sqrt{81 - 32} = 2\sqrt{49} = 14$.

**4** We can deduce that

$$\mathrm{p}(x) - 1 = (x-1)^5 \times \mathrm{q}(x) \qquad (*)$$

where $\mathrm{q}(x)$ is a quartic.

**(i)** Using $(*)$ we have $\mathrm{p}(1) - 1 = (1-1)^5 \times \mathrm{q}(1)$, so $\mathrm{p}(1) - 1 = 0 \implies \mathrm{p}(1) = 1$.

**(ii)** Differentiating $(*)$ gives:

$$\begin{aligned}\mathrm{p}'(x) &= 5(x-1)^4 \times \mathrm{q}(x) + (x-1)^5 \times \mathrm{q}'(x)\\ &= (x-1)^4\big(5\mathrm{q}(x) + (x-1)\mathrm{q}'(x)\big)\end{aligned}$$

and therefore $\mathrm{p}'(x)$ is divisible by $(x-1)^4$.

**(iii)** In a similar way to before we can write $\mathrm{p}(x) + 1 = (x+1)^5 \times \mathrm{q}_2(x)$. Substituting in $x = -1$ gives $\mathrm{p}(-1) = -1$ and differentiating can be used to show that $\mathrm{p}'(x)$ is divisible by $(x+1)^4$.

We can now write $\mathrm{p}'(x) = k(x-1)^4(x+1)^4 = k\big(x^2-1\big)^4 = k\big(x^8 - 4x^6 + 6x^4 - 4x^2 + 1\big)$.

Integrating gives:

$$\mathrm{p}(x) = k\big(\tfrac{1}{9}x^9 - \tfrac{4}{7}x^7 + \tfrac{6}{5}x^5 - \tfrac{4}{3}x^3 + x\big) + c$$

and using $\mathrm{p}(1) = 1$ and $\mathrm{p}(-1) = -1$ we have:

$$\begin{aligned}k\big(\tfrac{1}{9} - \tfrac{4}{7} + \tfrac{6}{5} - \tfrac{4}{3} + 1\big) + c &= 1\\ k\big(-\tfrac{1}{9} + \tfrac{4}{7} - \tfrac{6}{5} + \tfrac{4}{3} - 1\big) + c &= -1\end{aligned}$$

Adding these two together gives $c = 0$ and then the first one gives us:

$$k\big(\tfrac{35}{315} - \tfrac{180}{315} + \tfrac{378}{315} - \tfrac{420}{315} + \tfrac{315}{315}\big) = 1 \implies k = \tfrac{315}{128}\,.$$

So $\mathrm{p}(x) = \frac{315}{128}\big(\frac{1}{9}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\big)$.

**5** $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$ and $F_{10} = 55$.

**(i)** We have $F_i = F_{i-1} + F_{i-2}$ and, as long as $i \geqslant 4$, we have $F_{i-2} < F_{i-1}$ (when $i = 3$ we have equality as $F_1 = F_2$). Hence we have $F_i < 2F_{i-1}$ and so $\dfrac{1}{F_i} > \dfrac{1}{2F_{i-1}}$. We also have $\dfrac{1}{F_{i-1}} > \dfrac{1}{2F_{i-2}}$ (if $i \geqslant 5$) and so $\dfrac{1}{F_i} > \dfrac{1}{4F_{i-2}}$ etc.

We now have:

$$\begin{aligned}
S &= \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots \\
S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{2F_3} + \frac{1}{2F_4} + \frac{1}{2F_5} + \dots \\
S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{2F_3} + \frac{1}{4F_3} + \frac{1}{8F_3} + \dots \\
S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right)
\end{aligned}$$

Using the values of $F_1$, $F_2$, $F_3$ and the sum of an infinite GP gives us:

$$S > 1 + 1 + \frac{1}{2} \times 2 = 3\,.$$

In a similar way to above we have $F_i > 2F_{i-2}$ (for $i \geqslant 4$) and so $\dfrac{1}{F_i} < \dfrac{1}{2F_{i-2}}$. This is slightly different to before as you have to split up odd and even terms.

$$\begin{aligned}
S &= \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots \\
S &= \left(\frac{1}{F_1} + \frac{1}{F_2}\right) + \left(\frac{1}{F_4} + \frac{1}{F_6} + \frac{1}{F_8} + \dots\right) + \left(\frac{1}{F_3} + \frac{1}{F_5} + \frac{1}{F_7} + \dots\right) \\
S &< \left(\frac{1}{F_1} + \frac{1}{F_2}\right) + \left(\frac{1}{F_4} + \frac{1}{2F_4} + \frac{1}{4F_4} + \dots\right) + \left(\frac{1}{F_3} + \frac{1}{2F_3} + \frac{1}{4F_3} + \dots\right) \\
S &< 1 + 1 + \frac{1}{3} \times 2 + \frac{1}{2} \times 2 = 3\tfrac{2}{3}
\end{aligned}$$

**(ii)** For this part, use the same approach but take more terms before using the geometric series.

$$\begin{aligned}
S &= \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots \\
S &> \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5}\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right) \\
S &> 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 = 1 + 1 + \frac{37}{30}
\end{aligned}$$

Hence $S > 3\frac{7}{30} > 3\frac{6}{30} = 3.2$

And for the upper limit:

$$S = \frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4} + \frac{1}{F_5} + \frac{1}{F_6} + \dots$$
$$S = \left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4}\right) + \left(\frac{1}{F_5} + \frac{1}{F_7} + \dots\right) + \left(\frac{1}{F_6} + \frac{1}{F_8} + \dots\right)$$
$$S < \left(\frac{1}{F_1} + \frac{1}{F_2} + \frac{1}{F_3} + \frac{1}{F_4}\right) + \frac{1}{F_5}\left(1 + \frac{1}{2} + \dots\right) + \frac{1}{F_6}\left(1 + \frac{1}{2} + \dots\right)$$
$$S < 1 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} \times 2 + \frac{1}{8} \times 2$$

and so $S < 3\frac{29}{60} < 3\frac{1}{2}$.

**6** **(i)** Setting $a = 1$, $b = -1$, $x_n = x_{n+1} = x$ and $y_n = y_{n+1} = y$ gives the simultaneous equations:

$$\begin{aligned} x &= x^2 - y^2 + 1 \\ y &= 2xy + 1 \end{aligned}$$

Using the second equation we have $x = \dfrac{y-1}{2y}$, which we can substitute into the first equation to get:

$$\begin{aligned} \frac{y-1}{2y} &= \left(\frac{y-1}{2y}\right)^2 - y^2 + 1 \\ 2y(y-1) &= (y-1)^2 - 4y^2 \times y^2 + 4y^2 \\ 2y^2 - \cancel{2y} &= y^2 - \cancel{2y} + 1 - 4y^4 + 4y^2 \\ 4y^4 - 3y^2 - 1 &= 0 \\ (y-1)\left(4y^3 + 4y^2 + y + 1\right) &= 0 \\ (y-1)(y+1)\left(4y^2 + 1\right) &= 0 \end{aligned}$$

Therefore $y = 1$ or $y = -1$[1]. Using $x = \dfrac{y-1}{2y}$ gives the values as $(x_1, y_1) = (0, 1)$ and $(1, -1)$.

**(ii)** Taking $(x_1, y_1) = (-1, 1)$ gives $(x_2, y_2) = (a, b)$ and $(x_3, y_3) = (a^2 - b^2 + a, 2ab + b + 2)$. If the sequence is to have period 2 then we need $(x_1, y_1) = (x_3, y_3) \neq (x_2, y_2)$.

Using $(x_1, y_1) = (x_3, y_3)$ we have:

$$\begin{aligned} a^2 - b^2 + a &= -1 \\ 2ab + b + 2 &= 1 \end{aligned}$$

Similarly to before, we have $a = \dfrac{-b-1}{2b}$ and so:

$$\begin{aligned} \left(\frac{-b-1}{2b}\right)^2 - b^2 - \frac{b+1}{2b} &= -1 \\ (-b-1)^2 - 4b^4 - 2b(b+1) &= -4b^2 \\ b^2 + \cancel{2b} + 1 - 4b^4 - 2b^2 - \cancel{2b} &= -4b^2 \\ 4b^4 - 3b^2 - 1 &= 0 \end{aligned}$$

This last equation is identical to the one in $y$ for part **(i)**, so we have $b = \pm 1$. This gives $(a, b) = (0, -1)$ or $(-1, 1)$, but since the second gives $(x_2, y_2) = (x_1, y_1)$ (i.e. sequence is constant, not period 2) we discard that one to leave us with just one solution, $(a, b) = (0, -1)$.

There is a neat solution here, where you can spot that if you let $a = -x$ and $b = -y$ you get the same equations as in part **(i)**, which means you can deduce the values of $a$ and $b$ without solving the simultaneous equations.

[1] The equation $4y^2 = -1$ has no real solutions, and since we want $(x, y)$ to be a point in the Cartesian plane, we want $y$ to be real.

**7** **(i)** The binomial expansion gives:

$$\left(1+\frac{k}{100}\right)^{\frac{1}{2}} = 1+\frac{1}{2}\times\left(\frac{k}{100}\right)+\frac{1}{2!}\times\frac{1}{2}\times\frac{-1}{2}\times\left(\frac{k}{100}\right)^2 + \frac{1}{3!}\times\frac{1}{2}\times\frac{-1}{2}\times\frac{-3}{2}\times\left(\frac{k}{100}\right)^3+\dots$$
$$\approx 1+\frac{k}{200}-\frac{k^2}{80\,000}+\frac{k^3}{16\,000\,000}$$

**(a)** Substituting $k=8$ gives:

$$\begin{aligned}\left(1+\frac{k}{100}\right)^{\frac{1}{2}} &= \left(\frac{108}{100}\right)^{\frac{1}{2}}\\ &= \left(\frac{3\times 36}{100}\right)^{\frac{1}{2}}\\ &= \frac{6}{10}\times\sqrt{3}\end{aligned}$$

Using the binomial expansion (with $k=8$) gives:

$$\begin{aligned}\frac{6}{10}\times\sqrt{3} &\approx 1+\frac{8}{200}-\frac{8\times 8}{80\,000}+\frac{8\times 8\times 8}{16\,000\,000}\\ &= 1+\frac{4}{100}-\frac{8}{10\,000}+\frac{4\times 8}{1\,000\,000}\\ &= 1.040032-0.0008\\ &= 1.039232\end{aligned}$$

Then the approximation for $\sqrt{3}$ is given by:

$$\begin{aligned}\sqrt{3} &\approx \frac{1.039232}{0.6}\\ &= \frac{10.39232}{6}\end{aligned}$$

Carrying out the division gives $\sqrt{3}\approx 1.73205$.

**(b)** Here we need to find a suitable value of $k$, and remember that we want $k$ to be small in order to get a good approximation. Comparing to the previous part, we want $100+k=a^2\times 6$. Starting with $a=3$ we have:

$$\begin{aligned}3^2\times 6 = 54 &\implies k=-46\\ 4^2\times 6 = 96 &\implies k=-4\\ 5^2\times 6 = 150 &\implies k=50\end{aligned}$$

$k = 50$ is not a great choice as it is not small, so take $k = -4$.
Substituting $k = -4$ gives:

$$\begin{aligned}\left(\frac{96}{100}\right)^{\frac{1}{2}} &= \frac{4}{10} \times \sqrt{6} \\ &\approx 1 - \frac{4}{200} - \frac{4 \times 4}{80\,000} - \frac{4 \times 4 \times 4}{16\,000\,000} \\ &= 1 - \frac{2}{100} - \frac{2}{10\,000} - \frac{4}{1\,000\,000} \\ &= 1 - 0.020204 \\ &= 0.979796\end{aligned}$$

And so we have $\sqrt{6} \approx 9.79796 \div 4$, i.e. $\sqrt{6} \approx 2.44949$.

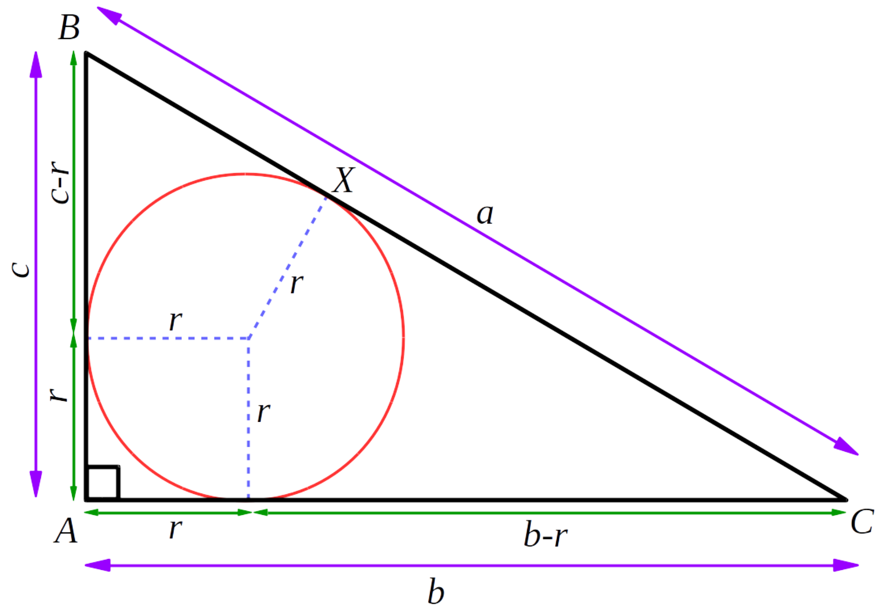
**(ii)** The first two terms of the binomial expansion of $\left(1 + \dfrac{k}{1000}\right)^{\frac{1}{3}}$ gives us:

$$\left(1 + \frac{k}{1000}\right)^{\frac{1}{3}} \approx 1 + \frac{k}{3000}\,.$$

By comparing to the previous part, we want to find a value of $k$ so that $1000+k = a^3 \times 3$, where $k$ is small compared to 1000. The value of $a$ which gives the smallest $k$ is $a = 7$, which gives $1000 + k = 1029 \implies k = 29$. We then have:

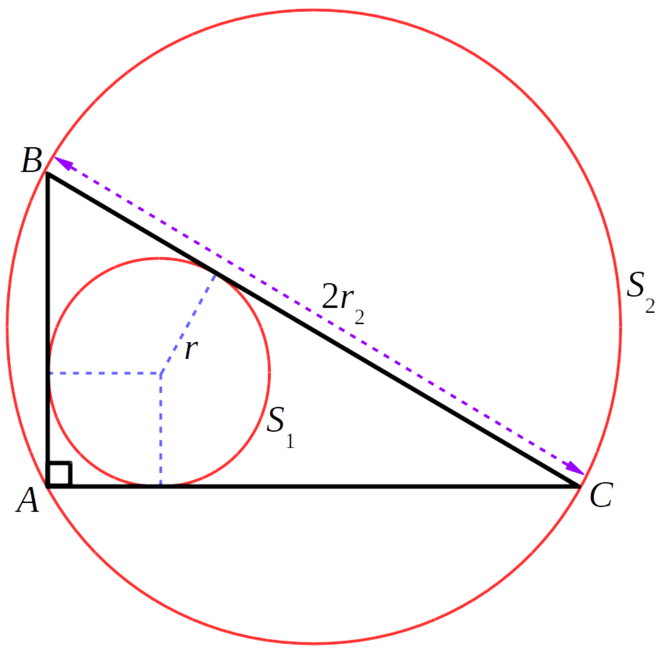
$$\begin{aligned}\left(\frac{1029}{1000}\right)^{\frac{1}{3}} &\approx 1 + \frac{29}{3000} \\ \frac{7}{10} \times \sqrt[3]{3} &\approx \frac{3029}{3000} \\ \sqrt[3]{3} &\approx \frac{3029}{300\not{0}} \times \frac{1\not{0}}{7} && \text{and so} \\ \sqrt[3]{3} &\approx \frac{3029}{2100} && \text{as required.}\end{aligned}$$

**8** You should start by drawing a large and clear diagram, maybe something like below:




Then we have $CX = b-r$ and $BX = c-r$ (adjacent sides of a kite). Hence $a = (b-r)+(c-r)$ and so $2r = b+c-a$.

We now have another diagram with the circumcircle shown as well as the incircle.




Since $\triangle ABC$ is right-angled, $BC$ is a diameter of $S_2$. Hence the radius of $S_2$ satisfies $2r_2 = a$.

The area of $S_2$ is $\pi r_2^2$ and the area between $S_1$ and the triangle is $\frac{1}{2}bc - \pi r^2$. Using the given fact about the ratio of these we have:

$$R\pi r_2^2 = \tfrac{1}{2}bc - \pi r^2$$

The result we are trying to obtain does not contain $r$ or $r_2$, so we substitute for these.

$$
\begin{aligned}
R\pi\left(\frac{a}{2}\right)^2 &= \tfrac{1}{2}bc - \pi\left(\frac{b+c-a}{2}\right)^2 \\
R\pi &= \frac{2bc}{a^2} - \pi\left(\frac{b+c-a}{a}\right)^2 \\
R\pi &= \frac{2bc}{a^2} - \pi\left(q-1\right)^2
\end{aligned}
$$

This is starting to look promising, but the $\frac{2bc}{a^2}$ needs writing in terms of $q$. We have:

$$
\begin{aligned}
q^2 &= \left(\frac{b+c}{a}\right)^2 \\
&= \frac{b^2+c^2+2bc}{a^2} \\
&= \frac{a^2+2bc}{a^2} \qquad \text{using Pythagoras' theorem} \\
&= 1 + \frac{2bc}{a^2}
\end{aligned}
$$

So we now have:

$$
\begin{aligned}
R\pi &= q^2 - 1 - \pi\left(q-1\right)^2 \\
&= q^2 - 1 - \pi(q^2 - 2q + 1) \\
&= q^2 - 1 - \pi q^2 + 2\pi q - \pi \\
&= -(\pi-1)q^2 + 2\pi q - (\pi+1) \qquad \text{as required}
\end{aligned}
$$

Note that this is a quadratic in $q$, and will have a maximum when $\frac{\mathrm{d}}{\mathrm{d}q}(\pi R) = 0$, i.e. when $q = \frac{2\pi}{2(\pi-1)} = \frac{\pi}{(\pi-1)}$. This gives

$$
\begin{aligned}
\pi R_{\max} &= -(\pi-1)\left(\frac{\pi}{\pi-1}\right)^2 + 2\pi\left(\frac{\pi}{\pi-1}\right) - (\pi+1) \\
&= -\frac{\pi^2}{\pi-1} + \frac{2\pi^2}{\pi-1} - (\pi+1) \\
&= \frac{\pi^2}{\pi-1} - \frac{(\pi+1)(\pi-1)}{\pi-1} \\
&= \frac{\pi^2 - \pi^2 + 1}{\pi-1} \\
&= \frac{1}{\pi-1}
\end{aligned}
$$

And so since $R_{\max} = \frac{1}{\pi(\pi-1)}$ we have $R \leqslant \frac{1}{\pi(\pi-1)}$.

**9** **(i)** One stumbling block is not reading all the information in the "stem"! Since you are told that $\lambda = 1 + \sqrt{2}$ you know that $\lambda - 1 = \sqrt{2}$ etc.

We have:

$$\begin{aligned}
\sum_{r=0}^{n} b_r &= \left(\lambda^0 - \mu^0\right) + \left(\lambda^1 - \mu^1\right) + \left(\lambda^2 - \mu^2\right) + \ldots + \left(\lambda^n - \mu^n\right) \\
&= \left(1 + \lambda^1 + \lambda^2 + \ldots + \lambda^n\right) - \left(1 + \mu^1 + \mu^2 + \ldots \mu^n\right) \\
&= \frac{\lambda^{n+1} - 1}{\lambda - 1} - \frac{\mu^{n+1} - 1}{\mu - 1} \\
&= \frac{\lambda^{n+1} - 1}{\sqrt{2}} - \frac{\mu^{n+1} - 1}{-\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\left(\lambda^{n+1} + \mu^{n+1}\right) - 2 \times \frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} a_{n+1} - \sqrt{2}
\end{aligned}$$

Similarly:

$$\begin{aligned}
\sum_{r=0}^{n} a_r &= \left(\lambda^0 + \mu^0\right) + \left(\lambda^1 + \mu^1\right) + \left(\lambda^2 + \mu^2\right) + \ldots + \left(\lambda^n + \mu^n\right) \\
&= \left(1 + \lambda^1 + \lambda^2 + \ldots + \lambda^n\right) + \left(1 + \mu^1 + \mu^2 + \ldots \mu^n\right) \\
&= \frac{\lambda^{n+1} - 1}{\lambda - 1} + \frac{\mu^{n+1} - 1}{\mu - 1} \\
&= \frac{\lambda^{n+1} - 1}{\sqrt{2}} + \frac{\mu^{n+1} - 1}{-\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}\left(\lambda^{n+1} - \mu^{n+1}\right) \\
&= \frac{1}{\sqrt{2}} b_{n+1}
\end{aligned}$$

**(ii)** Here we have a "nested sum". Start by evaluating the "inner sum".

$$\begin{aligned}
\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) &= \sum_{m=0}^{2n}\left(\frac{1}{\sqrt{2}} b_{m+1}\right) \\
&= \sum_{m=0}^{2n+1}\left(\frac{1}{\sqrt{2}} b_m\right) \qquad \text{since } b_0 = 0 \\
&= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} a_{(2n+1)+1} - \sqrt{2}\right) \\
&= \frac{1}{2}\left(a_{2n+2} - 2\right) \\
&= \frac{1}{2}\left(\lambda^{2n+2} + \mu^{2n+2} - 2\right)
\end{aligned}$$

We are trying to get something like $(b_{n+1})^2$.[2] Noting that $\lambda \times \mu = 1 - 2 = -1$, and using the fact that $n$ is odd, so that $n+1$ is even and $(\lambda\mu)^{n+1} = 1$ we have:

$$\begin{aligned}\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) &= \frac{1}{2}\left(\lambda^{2n+2} + \mu^{2n+2} - 2\right)\\ &= \frac{1}{2}\left(\left(\lambda^{n+1}\right)^2 + \left(\mu^{n+1}\right)^2 - 2\left(\lambda\mu\right)^{n+1}\right)\\ &= \frac{1}{2}\left(\lambda^{n+1} - \mu^{n+1}\right)^2\\ &= \frac{1}{2}\left(b_{n+1}\right)^2 .\end{aligned}$$

When $n$ is even, $n+1$ is odd and $(\lambda\mu)^{n+1} = -1$. Then we have:

$$\begin{aligned}\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) &= \frac{1}{2}\left(\lambda^{2n+2} + \mu^{2n+2} - 2\right)\\ &= \frac{1}{2}\left(\left(\lambda^{n+1}\right)^2 + \left(\mu^{n+1}\right)^2 + 2\left(\lambda\mu\right)^{n+1}\right)\\ &= \frac{1}{2}\left(\lambda^{n+1} + \mu^{n+1}\right)^2\\ &= \frac{1}{2}\left(a_{n+1}\right)^2 .\end{aligned}$$

**(iii)** From part **(i)** we have $\left(\sum_{r=0}^{n} a_r\right)^2 = \left(\frac{1}{\sqrt{2}}b_{n+1}\right)^2 = \frac{1}{2}\left(b_{n+1}\right)^2$.

We also need:

$$\begin{aligned}\sum_{r=0}^{n} a_{2r+1} &= a_1 + a_3 + a_5 + \ldots + a_{2n+1}\\ &= \left(\lambda^1 + \lambda^3 + \lambda^5 + \ldots + \lambda^{2n+1}\right) + \left(\mu^1 + \mu^3 + \mu^5 + \ldots + \mu^{2n+1}\right)\\ &= \lambda\left(1 + \lambda^2 + \left(\lambda^2\right)^2 + \ldots + \left(\lambda^2\right)^n\right) + \mu\left(1 + \mu^2 + \left(\mu^2\right)^2 + \ldots + \left(\mu^2\right)^n\right)\\ &= \frac{\lambda\left(\left(\lambda^2\right)^{n+1} - 1\right)}{\lambda^2 - 1} + \frac{\mu\left(\left(\mu^2\right)^{n+1} - 1\right)}{\mu^2 - 1}\end{aligned}$$

[2] What I did, and what I would expect a lot of people to do, is expand $(b_{n+1})^2 = \left(\lambda^{n+1} - \mu^{n+1}\right)^2$ and try and figure out how it is related to what I have already done. Then I wrote up a solution "going the correct way".

$\lambda^2 - 1 = 3 + 2\sqrt{2} - 1 = 2(1+\sqrt{2}) = 2\lambda$, and similarly $\mu^2 - 1 = 2\mu$. We now have:

$$\begin{aligned}\sum_{r=0}^{n} a_{2r+1} &= \frac{\lambda\left(\left(\lambda^2\right)^{n+1} - 1\right)}{\lambda^2 - 1} + \frac{\mu\left(\left(\mu^2\right)^{n+1} - 1\right)}{\mu^2 - 1} \\ &= \frac{\cancel{\lambda}\left(\left(\lambda^2\right)^{n+1} - 1\right)}{2\cancel{\lambda}} + \frac{\cancel{\mu}\left(\left(\mu^2\right)^{n+1} - 1\right)}{2\cancel{\mu}} \\ &= \frac{1}{2}\left(\left(\lambda^{n+1}\right)^2 + \left(\mu^{n+1}\right)^2 - 2\right)\end{aligned}$$

Which — as in part **(ii)**— is equal to $\frac{1}{2}\left(b_{n+1}\right)^2$ if $n$ is odd and $\frac{1}{2}\left(a_{n+1}\right)^2$ if $n$ is even.

$$\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = \tfrac{1}{2}\left(b_{n+1}\right)^2 - \tfrac{1}{2}\left(b_{n+1}\right)^2 = 0 \qquad \text{if } n \text{ is odd}$$

$$\begin{aligned}\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} &= \tfrac{1}{2}\left(b_{n+1}\right)^2 - \tfrac{1}{2}\left(a_{n+1}\right)^2 \\ &= \tfrac{1}{2}\left(\left(\lambda^{n+1}\right)^2 + \left(\mu^{n+1}\right)^2 - 2\left(\lambda\mu\right)^{n+1}\right) \\ &\qquad - \tfrac{1}{2}\left(\left(\lambda^{n+1}\right)^2 + \left(\mu^{n+1}\right)^2 + 2\left(\lambda\mu\right)^{n+1}\right) \\ &= -2\left(\lambda\mu\right)^{n+1} \\ &= -2\times(-1)^{n+1} = 2 \qquad \text{if } n \text{ is even.}\end{aligned}$$