

Section A: Pure Mathematics

- 1 Sketch on the same axes the two curves C_1 and C_2 , given by

$$\begin{aligned} C_1: \quad xy &= 1, \\ C_2: \quad x^2 - y^2 &= 2. \end{aligned}$$

The curves intersect at P and Q . Given that the coordinates of P are (a, b) (which you need not evaluate), write down the coordinates of Q in terms of a and b .

The tangent to C_1 through P meets the tangent to C_2 through Q at the point M , and the tangent to C_2 through P meets the tangent to C_1 through Q at N . Show that the coordinates of M are $(-b, a)$ and write down the coordinates of N .

Show that $PMQN$ is a square.

- 2 Use the substitution $x = 2 - \cos \theta$ to evaluate the integral

$$\int_{3/2}^2 \left(\frac{x-1}{3-x} \right)^{\frac{1}{2}} dx.$$

Show that, for $a < b$,

$$\int_p^q \left(\frac{x-a}{b-x} \right)^{\frac{1}{2}} dx = \frac{(b-a)(\pi + 3\sqrt{3-6})}{12},$$

where $p = (3a + b)/4$ and $q = (a + b)/2$.

- 3 Given that $\alpha = e^{i\pi/3}$, prove that $1 + \alpha^2 = \alpha$.

A triangle in the Argand plane has vertices A , B , and C represented by the complex numbers p , $q\alpha^2$ and $-r\alpha$ respectively, where p , q and r are positive real numbers. Sketch the triangle ABC .

Three equilateral triangles ABL , BCM and CAN (each lettered clockwise) are erected on sides AB , BC and CA respectively. Show that the complex number representing N is $(1 - \alpha)p - \alpha^2 r$ and find similar expressions for the complex numbers representing L and M .

Show that lines LC , MA and NB all meet at the origin, and that these three line segments have the common length $p + q + r$.

4 The function $f(x)$ is defined by

$$f(x) = \frac{x(x-2)(x-a)}{x^2-1}.$$

Prove algebraically that the line $y = x + c$ intersects the curve $y = f(x)$ if $|a| \geq 1$, but there are values of c for which there are no points of intersection if $|a| < 1$.

Find the equation of the oblique asymptote of the curve $y = f(x)$. Sketch the graph in the two cases (i) $a < -1$; and (ii) $-1 < a < -\frac{1}{2}$. (You need not calculate the turning points.)

5 Given two non-zero vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ define $\Delta(\mathbf{a}, \mathbf{b})$ by $\Delta(\mathbf{a}, \mathbf{b}) = a_1b_2 - a_2b_1$.

Let A, B and C be points with position vectors \mathbf{a}, \mathbf{b} and \mathbf{c} , respectively, no two of which are parallel. Let P, Q and R be points with position vectors \mathbf{p}, \mathbf{q} and \mathbf{r} , respectively, none of which are parallel.

(i) Show that there exists a 2×2 matrix \mathbf{M} such that P and Q are the images of A and B under the transformation represented by \mathbf{M} .

(ii) Show that $\Delta(\mathbf{a}, \mathbf{b}) \mathbf{c} + \Delta(\mathbf{c}, \mathbf{a}) \mathbf{b} + \Delta(\mathbf{b}, \mathbf{c}) \mathbf{a} = \mathbf{0}$.

Hence, or otherwise, prove that a necessary and sufficient condition for the points P, Q , and R to be the images of points A, B and C under the transformation represented by some 2×2 matrix \mathbf{M} is that

$$\Delta(\mathbf{a}, \mathbf{b}) : \Delta(\mathbf{b}, \mathbf{c}) : \Delta(\mathbf{c}, \mathbf{a}) = \Delta(\mathbf{p}, \mathbf{q}) : \Delta(\mathbf{q}, \mathbf{r}) : \Delta(\mathbf{r}, \mathbf{p}).$$

6 Given that

$$x^4 + px^2 + qx + r = (x^2 - ax + b)(x^2 + ax + c),$$

express p, q and r in terms of a, b and c .

Show also that a^2 is a root of the cubic equation

$$u^3 + 2pu^2 + (p^2 - 4r)u - q^2 = 0.$$

Explain why this equation always has a non-negative root, and verify that $u = 9$ is a root in the case $p = -1, q = -6, r = 15$.

Hence, or otherwise, express

$$y^4 - 8y^3 + 23y^2 - 34y + 39$$

as a product of two quadratic factors.

7 Given that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{r!} + \cdots,$$

use the binomial theorem to show that

$$\left(1 + \frac{1}{n}\right)^n < e$$

for any positive integer n .

The product $P(n)$ is defined, for any positive integer n , by

$$P(n) = \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{9}{8} \cdot \cdots \cdot \frac{2^n + 1}{2^n}.$$

Use the arithmetic-geometric mean inequality,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{\frac{1}{n}},$$

to show that $P(n) < e$ for all n .

Explain briefly why $P(n)$ tends to a limit as $n \rightarrow \infty$. Show that this limit, L , satisfies $2 < L \leq e$.

8 The sequence a_n is defined by $a_0 = 1$, $a_1 = 1$, and

$$a_n = \frac{1 + a_{n-1}^2}{a_{n-2}} \quad (n \geq 2).$$

Prove by induction that

$$a_n = 3a_{n-1} - a_{n-2} \quad (n \geq 2).$$

Hence show that

$$a_n = \frac{\alpha^{2n-1} + \alpha^{-(2n-1)}}{\sqrt{5}} \quad (n \geq 1),$$

where $\alpha = \frac{1 + \sqrt{5}}{2}$.

Section B: Mechanics

- 9 Two small discs of masses m and μm lie on a smooth horizontal surface. The disc of mass μm is at rest, and the disc of mass m is projected towards it with velocity \mathbf{u} . After the collision, the disc of mass μm moves in the direction given by unit vector \mathbf{n} . The collision is perfectly elastic.

(i) Show that the speed of the disc of mass μm after the collision is $\frac{2\mathbf{u} \cdot \mathbf{n}}{1 + \mu}$.

- (ii) Given that the two discs have equal kinetic energy after the collision, find an expression for the cosine of the angle between \mathbf{n} and \mathbf{u} and show that $3 - \sqrt{8} \leq \mu \leq 3 + \sqrt{8}$.

- 10 A sphere of radius a and weight W rests on horizontal ground. A thin uniform beam of weight $3\sqrt{3}W$ and length $2a$ is freely hinged to the ground at X , which is a distance $\sqrt{3}a$ from the point of contact of the sphere with the ground. The beam rests on the sphere, lying in the same vertical plane as the centre of the sphere. The coefficients of friction between the beam and the sphere and between the sphere and the ground are μ_1 and μ_2 respectively.

Given that the sphere is on the point of slipping at its contacts with both the ground and the beam, find the values of μ_1 and μ_2 .

- 11 A thin beam is fixed at a height $2a$ above a horizontal plane. A uniform straight rod ACB of length $9a$ and mass m is supported by the beam at C . Initially, the rod is held so that it is horizontal and perpendicular to the beam. The distance AC is $3a$, and the coefficient of friction between the beam and the rod is μ .

The rod is now released. Find the minimum value of μ for which B strikes the horizontal plane before slipping takes place at C .

Section C: Probability and Statistics

- 12** In a lottery, any one of N numbers, where N is large, is chosen at random and independently for each player by machine. Each week there are $2N$ players and one winning number is drawn. Write down an exact expression for the probability that there are three or fewer winners in a week, given that you hold a winning ticket that week. Using the fact that

$$\left(1 - \frac{a}{n}\right)^n \approx e^{-a}$$

for n much larger than a , or otherwise, show that this probability is approximately $\frac{2}{3}$.

Discuss briefly whether this probability would increase or decrease if the numbers were chosen by the players.

Show that the expected number of winners in a week, given that you hold a winning ticket that week, is $3 - N^{-1}$.

- 13** A set of n dice is rolled repeatedly. For each die the probability of showing a six is p . Show that the probability that the first of the dice to show a six does so on the r th roll is

$$q^{nr}(q^{-n} - 1)$$

where $q = 1 - p$.

Determine, and simplify, an expression for the probability generating function for this distribution, in terms of q and n . The first of the dice to show a six does so on the R th roll. Find the expected value of R and show that, in the case $n = 2$, $p = 1/6$, this value is $36/11$.

Show that the probability that the last of the dice to show a six does so on the r th roll is

$$(1 - q^r)^n - (1 - q^{r-1})^n.$$

Find, for the case $n = 2$, the probability generating function. The last of the dice to show a six does so on the S th roll. Find the expected value of S and evaluate this when $p = 1/6$.

- 14** The random variable X takes only the values x_1 and x_2 (where $x_1 \neq x_2$), and the random variable Y takes only the values y_1 and y_2 (where $y_1 \neq y_2$). Their joint distribution is given by

$$P(X = x_1, Y = y_1) = a; \quad P(X = x_1, Y = y_2) = q - a; \quad P(X = x_2, Y = y_1) = p - a.$$

Show that if $E(XY) = E(X)E(Y)$ then

$$(a - pq)(x_1 - x_2)(y_1 - y_2) = 0.$$

Hence show that two random variables each taking only two distinct values are independent if $E(XY) = E(X)E(Y)$.

Give a joint distribution for two random variables A and B , each taking the three values -1 , 0 and 1 with probability $\frac{1}{3}$, which have $E(AB) = E(A)E(B)$, but which are not independent.