Section A: Pure Mathematics

1 Show that the equation of any circle passing through the points of intersection of the ellipse $(x+2)^2 + 2y^2 = 18$ and the ellipse $9(x-1)^2 + 16y^2 = 25$ can be written in the form

$$x^2 - 2ax + y^2 = 5 - 4a\ .$$

2 Let $\mathrm{f}(x) = x^m(x-1)^n$, where m and n are both integers greater than 1. Show that the curve $y = \mathrm{f}(x)$ has a stationary point with $0 < x < 1$. By considering $\mathrm{f}''(x)$, show that this stationary point is a maximum if n is even and a minimum if n is odd.

Sketch the graphs of $\mathrm{f}(x)$ in the four cases that arise according to the values of m and n.

3 Show that $(a+b)^2 \leqslant 2a^2 + 2b^2$.

Find the stationary points on the curve $y = \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\frac{1}{2}} + \left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{1}{2}}$, where a and b are constants. State, with brief reasons, which points are maxima and which are minima. Hence prove that

$$|a| + |b| \leqslant \left(a^2\cos^2\theta + b^2\sin^2\theta\right)^{\frac{1}{2}} + \left(a^2\sin^2\theta + b^2\cos^2\theta\right)^{\frac{1}{2}} \leqslant \left(2a^2 + 2b^2\right)^{\frac{1}{2}}\ .$$

4 Give a sketch of the curve $y = \dfrac{1}{1+x^2}$, for $x \geqslant 0$.

Find the equation of the line that intersects the curve at $x = 0$ and is tangent to the curve at some point with $x > 0$. Prove that there are no further intersections between the line and the curve. Draw the line on your sketch.

By considering the area under the curve for $0 \leqslant x \leqslant 1$, show that $\pi > 3$.

Show also, by considering the volume formed by rotating the curve about the y axis, that $\ln 2 > 2/3$.

[**Note**: $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}$.]

5 Let

$$\mathrm{f}(x) = x^n + a_1 x^{n-1} + \cdots + a_n \ ,$$

where a_1, a_2, $\ldots$, a_n are given numbers. It is given that $\mathrm{f}(x)$ can be written in the form

$$\mathrm{f}(x) = (x + k_1)(x + k_2) \cdots (x + k_n) \ .$$

By considering $\mathrm{f}(0)$, or otherwise, show that $k_1 k_2 \ldots k_n = a_n$.

Show also that

$$(k_1 + 1)(k_2 + 1) \cdots (k_n + 1) = 1 + a_1 + a_2 + \cdots + a_n$$

and give a corresponding result for $(k_1 - 1)(k_2 - 1) \cdots (k_n - 1)$.

Find the roots of the equation

$$x^4 + 22x^3 + 172x^2 + 552x + 576 = 0 \ ,$$

given that they are all integers.

6 A pyramid stands on horizontal ground. Its base is an equilateral triangle with sides of length a, the other three sides of the pyramid are of length b and its volume is V. Given that the formula for the volume of any pyramid is $\frac{1}{3} \times$ area of base $\times$ height, show that

$$V = \frac{1}{12} a^2 (3b^2 - a^2)^{\frac{1}{2}} \ .$$

The pyramid is then placed so that a non-equilateral face lies on the ground. Show that the new height, h, of the pyramid is given by

$$h^2 = \frac{a^2 (3b^2 - a^2)}{4b^2 - a^2} \ .$$

Find, in terms of a and b, the angle between the equilateral triangle and the horizontal.

7 Let

$$I = \int_0^a \frac{\cos x}{\sin x + \cos x} \, \mathrm{d}x \quad \text{and} \quad J = \int_0^a \frac{\sin x}{\sin x + \cos x} \, \mathrm{d}x \ ,$$

where $0 \leqslant a < \frac{3}{4}\pi$. By considering $I + J$ and $I - J$, show that $2I = a + \ln(\sin a + \cos a)$.

Find also:

(i) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x}{p \sin x + q \cos x} \, \mathrm{d}x$, where p and q are positive numbers;

(ii) $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\cos x + 4}{3 \sin x + 4 \cos x + 25} \, \mathrm{d}x$.

8 I borrow C pounds at interest rate $100\alpha\,\%$ per year. The interest is added at the end of each year. Immediately after the interest is added, I make a repayment. The amount I repay at the end of the kth year is R_k pounds and the amount I owe at the beginning of kth year is C_k pounds (with $C_1 = C$). Express C_{n+1} in terms of R_k ($k = 1, 2, \ldots, n$), α and C and show that, if I pay off the loan in N years with repayments given by $R_k = (1+\alpha)^k r$, where r is constant, then $r = C/N$.

If instead I pay off the loan in N years with N equal repayments of R pounds, show that

$$\frac{R}{C} = \frac{\alpha(1+\alpha)^N}{(1+\alpha)^N - 1},$$

and that $R/C \approx 27/103$ in the case $\alpha = 1/50$, $N = 4$.

Section B: Mechanics

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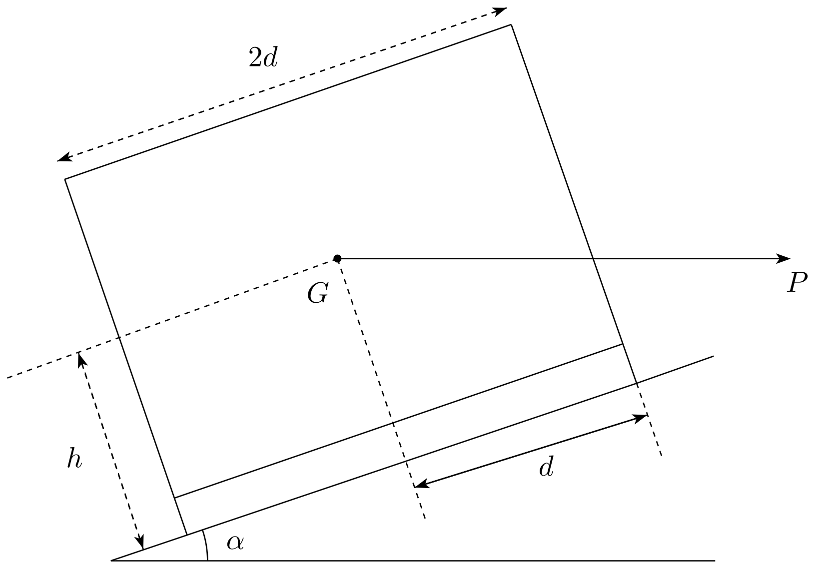

A lorry of weight W stands on a plane inclined at an angle α to the horizontal. Its wheels are a distance $2d$ apart, and its centre of gravity G is at a distance h from the plane, and halfway between the sides of the lorry. A horizontal force P acts on the lorry through G, as shown.

(i) If the normal reactions on the lower and higher wheels of the lorry are equal, show that the sum of the frictional forces between the wheels and the ground is zero.

(ii) If P is such that the lorry does not tip over (but the normal reactions on the lower and higher wheels of the lorry need not be equal), show that

$$W \tan(\alpha - \beta) \leqslant P \leqslant W \tan(\alpha + \beta) \ ,$$

where $\tan \beta = d/h$.

10 A bicycle pump consists of a cylinder and a piston. The piston is pushed in with steady speed u. A particle of air moves to and fro between the piston and the end of the cylinder, colliding perfectly elastically with the piston and the end of the cylinder, and always moving parallel with the axis of the cylinder. Initially, the particle is moving towards the piston at speed v. Show that the speed, v_n, of the particle just after the nth collision with the piston is given by $v_n = v + 2nu$.

Let d_n be the distance between the piston and the end of the cylinder at the nth collision, and let t_n be the time between the nth and $(n+1)$th collisions. Express $d_n - d_{n+1}$ in terms of u and t_n, and show that

$$d_{n+1} = \frac{v + (2n-1)u}{v + (2n+1)u}\, d_n \ .$$

Express d_n in terms of d_1, u, v and n.

In the case $v = u$, show that $ut_n = \dfrac{d_1}{n(n+1)}$.

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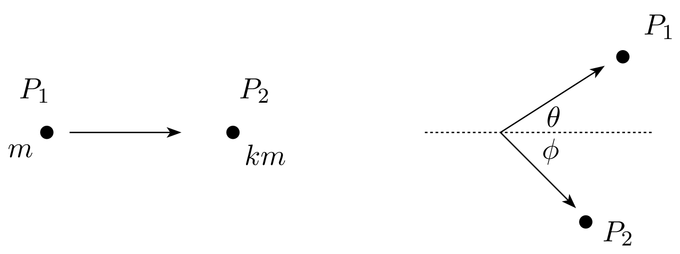

A particle P_1 of mass m collides with a particle P_2 of mass km which is at rest. No energy is lost in the collision. The direction of motion of P_1 and P_2 after the collision make non-zero angles of θ and ϕ, respectively, with the direction of motion of P_1 before the collision, as shown. Show that

$$\sin^2\theta + k\sin^2\phi = k\sin^2(\theta+\phi) \ .$$

Show that, if the angle between the particles after the collision is a right angle, then $k = 1$.

Section C: Probability and Statistics

12 Harry the Calculating Horse will do any mathematical problem I set him, providing the answer is 1, 2, 3 or 4. When I set him a problem, he places a hoof on a large grid consisting of unit squares and his answer is the number of squares partly covered by his hoof. Harry has circular hoofs, of radius $1/4$ unit.

After many years of collaboration, I suspect that Harry no longer bothers to do the calculations, instead merely placing his hoof on the grid completely at random. I often ask him to divide 4 by 4, but only about $1/4$ of his answers are right; I often ask him to add 2 and 2, but disappointingly only about $\pi/16$ of his answers are right. Is this consistent with my suspicions?

I decide to investigate further by setting Harry many problems, the answers to which are 1, 2, 3, or 4 with equal frequency. If Harry is placing his hoof at random, find the expected value of his answers. The average of Harry's answers turns out to be 2. Should I get a new horse?

13 The random variable U takes the values $+1, 0$ and -1, each with probability $\frac{1}{3}$. The random variable V takes the values $+1$ and -1 as follows:

if $U = 1$, then $\mathrm{P}(V = 1) = \frac{1}{3}$ and $\mathrm{P}(V = -1) = \frac{2}{3}$;
if $U = 0$, then $\mathrm{P}(V = 1) = \frac{1}{2}$ and $\mathrm{P}(V = -1) = \frac{1}{2}$;
if $U = -1$, then $\mathrm{P}(V = 1) = \frac{2}{3}$ and $\mathrm{P}(V = -1) = \frac{1}{3}$.

(i) Show that the probability that both roots of the equation $x^2 + Ux + V = 0$ are real is $\frac{1}{2}$.

(ii) Find the expected value of the larger root of the equation $x^2 + Ux + V = 0$, given that both roots are real.

(iii) Find the probability that the roots of the equation

$$x^3 + (U - 2V)x^2 + (1 - 2UV)x + U = 0$$

are all positive.

14 In order to get money from a cash dispenser I have to punch in an identification number. I have forgotten my identification number, but I do know that it is equally likely to be any one of the integers $1, 2, \ldots, n$. I plan to punch in integers in order until I get the right one. I can do this at the rate of r integers per minute. As soon as I punch in the first wrong number, the police will be alerted. The probability that they will arrive within a time t minutes is $1 - \mathrm{e}^{-\lambda t}$, where λ is a positive constant. If I follow my plan, show that the probability of the police arriving before I get my money is

$$\sum_{k=1}^{n} \frac{1 - \mathrm{e}^{-\lambda(k-1)/r}}{n} \; .$$

Simplify the sum.

On past experience, I know that I will be so flustered that I will just punch in possible integers at random, without noticing which I have already tried. Show that the probability of the police arriving before I get my money is

$$1 - \frac{1}{n - (n-1)\mathrm{e}^{-\lambda/r}} \; .$$