

Section A: Pure Mathematics

1 Show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{1}{1 - \cos 2\theta} d\theta = \frac{\sqrt{3}}{2} - \frac{1}{2}.$$

By using the substitution $x = \sin 2\theta$, or otherwise, show that

$$\int_{\sqrt{3}/2}^1 \frac{1}{1 - \sqrt{1 - x^2}} dx = \sqrt{3} - 1 - \frac{\pi}{6}.$$

Hence evaluate the integral

$$\int_1^{2/\sqrt{3}} \frac{1}{y(y - \sqrt{y^2 - 1^2})} dy.$$

2 Show that setting $z - z^{-1} = w$ in the quartic equation

$$z^4 + 5z^3 + 4z^2 - 5z + 1 = 0$$

results in the quadratic equation $w^2 + 5w + 6 = 0$. Hence solve the above quartic equation.

Solve similarly the equation

$$2z^8 - 3z^7 - 12z^6 + 12z^5 + 22z^4 - 12z^3 - 12z^2 + 3z + 2 = 0.$$

3 The n th Fermat number, F_n , is defined by

$$F_n = 2^{2^n} + 1, \quad n = 0, 1, 2, \dots,$$

where 2^{2^n} means 2 raised to the power 2^n . Calculate F_0 , F_1 , F_2 and F_3 . Show that, for $k = 1$, $k = 2$ and $k = 3$,

$$F_0 F_1 \dots F_{k-1} = F_k - 2. \quad (*)$$

Prove, by induction, or otherwise, that $(*)$ holds for all $k \geq 1$. Deduce that no two Fermat numbers have a common factor greater than 1.

Hence show that there are infinitely many prime numbers.

4 Give a sketch to show that, if $f(x) > 0$ for $p < x < q$, then $\int_p^q f(x)dx > 0$.

(i) By considering $f(x) = ax^2 - bx + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $3b < 2a + 6c$.

(ii) By considering $f(x) = a \sin^2 x - b \sin x + c$ show that, if $a > 0$ and $b^2 < 4ac$, then $4b < (a + 2c)\pi$.

(iii) Show that, if $a > 0$, $b^2 < 4ac$ and $q > p > 0$, then

$$b \ln(q/p) < a \left(\frac{1}{p} - \frac{1}{q} \right) + c(q - p).$$

5 The numbers x_n , where $n = 0, 1, 2, \dots$, satisfy

$$x_{n+1} = kx_n(1 - x_n).$$

(i) Prove that, if $0 < k < 4$ and $0 < x_0 < 1$, then $0 < x_n < 1$ for all n .

(ii) Given that $x_0 = x_1 = x_2 = \dots = a$, with $a \neq 0$ and $a \neq 1$, find k in terms of a .

(iii) Given instead that $x_0 = x_2 = x_4 = \dots = a$, with $a \neq 0$ and $a \neq 1$, show that $ab^3 - b^2 + (1 - a) = 0$, where $b = k(1 - a)$. Given, in addition, that $x_1 \neq a$, find the possible values of k in terms of a .

6 The lines l_1, l_2 and l_3 lie in an inclined plane P and pass through a common point A . The line l_2 is a line of greatest slope in P . The line l_1 is perpendicular to l_3 and makes an acute angle α with l_2 . The angles between the horizontal and l_1, l_2 and l_3 are $\pi/6, \beta$ and $\pi/4$, respectively. Show that $\cos \alpha \sin \beta = \frac{1}{2}$ and find the value of $\sin \alpha \sin \beta$. Deduce that $\beta = \pi/3$.

The lines l_1 and l_3 are rotated in P about A so that l_1 and l_3 remain perpendicular to each other. The new acute angle between l_1 and l_2 is θ . The new angles which l_1 and l_3 make with the horizontal are ϕ and 2ϕ , respectively. Show that

$$\tan^2 \theta = \frac{3 + \sqrt{13}}{2}.$$

- 7 In 3-dimensional space, the lines m_1 and m_2 pass through the origin and have directions $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$, respectively. Find the directions of the two lines m_3 and m_4 that pass through the origin and make angles of $\pi/4$ with both m_1 and m_2 . Find also the cosine of the acute angle between m_3 and m_4 .

The points A and B lie on m_1 and m_2 respectively, and are each at distance $\lambda\sqrt{2}$ units from O . The points P and Q lie on m_3 and m_4 respectively, and are each at distance 1 unit from O . If all the coordinates (with respect to axes \mathbf{i} , \mathbf{j} and \mathbf{k}) of A , B , P and Q are non-negative, prove that:

- (i) there are only two values of λ for which AQ is perpendicular to BP ;
- (ii) there are no non-zero values of λ for which AQ and BP intersect.

- 8 Find y in terms of x , given that:

$$\begin{aligned} \text{for } x < 0, \quad \frac{dy}{dx} &= -y \quad \text{and} \quad y = a \quad \text{when } x = -1; \\ \text{for } x > 0, \quad \frac{dy}{dx} &= y \quad \text{and} \quad y = b \quad \text{when } x = 1. \end{aligned}$$

Sketch a solution curve. Determine the condition on a and b for the solution curve to be continuous (that is, for there to be no 'jump' in the value of y) at $x = 0$.

Solve the differential equation

$$\frac{dy}{dx} = |e^x - 1| y$$

given that $y = e^e$ when $x = 1$ and that y is continuous at $x = 0$. Write down the following limits:

$$\text{(i)} \quad \lim_{x \rightarrow +\infty} y \exp(-e^x); \quad \text{(ii)} \quad \lim_{x \rightarrow -\infty} y e^{-x}.$$

Section B: Mechanics

- 9 A particle is projected from a point O on a horizontal plane with speed V and at an angle of elevation α . The vertical plane in which the motion takes place is perpendicular to two vertical walls, both of height h , at distances a and b from O . Given that the particle just passes over the walls, find $\tan \alpha$ in terms of a , b and h and show that

$$\frac{2V^2}{g} = \frac{ab}{h} + \frac{(a+b)^2 h}{ab}.$$

The heights of the walls are now increased by the same small positive amount δh . A second particle is projected so that it just passes over both walls, and the new angle and speed of projection are $\alpha + \delta\alpha$ and $V + \delta V$, respectively. Show that

$$\sec^2 \alpha \delta\alpha \approx \frac{a+b}{ab} \delta h,$$

and deduce that $\delta\alpha > 0$. Show also that δV is positive if $h > ab/(a+b)$ and negative if $h < ab/(a+b)$.

- 10 A competitor in a Marathon of $42\frac{3}{8}$ km runs the first t hours of the race at a constant speed of 13 km h^{-1} and the remainder at a constant speed of $14 + 2t/T \text{ km h}^{-1}$, where T hours is her time for the race. Show that the minimum possible value of T over all possible values of t is 3.

The speed of another competitor decreases linearly with respect to time from 16 km h^{-1} at the start of the race. If both of these competitors have a run time of 3 hours, find the maximum distance between them at any stage of the race.

- 11 A rigid straight beam AB has length l and weight W . Its weight per unit length at a distance x from B is $\alpha W l^{-1} (x/l)^{\alpha-1}$, where α is a positive constant. Show that the centre of mass of the beam is at a distance $\alpha l/(\alpha+1)$ from B .

The beam is placed with the end A on a rough horizontal floor and the end B resting against a rough vertical wall. The beam is in a vertical plane at right angles to the plane of the wall and makes an angle of θ with the floor. The coefficient of friction between the floor and the beam is μ and the coefficient of friction between the wall and the beam is also μ . Show that, if the equilibrium is limiting at both A and B , then

$$\tan \theta = \frac{1 - \alpha\mu^2}{(1 + \alpha)\mu}.$$

Given that $\alpha = 3/2$ and given also that the beam slides for any $\theta < \pi/4$ find the greatest possible value of μ .

Section C: Probability and Statistics

- 12 On K consecutive days each of L identical coins is thrown M times. For each coin, the probability of throwing a head in any one throw is p (where $0 < p < 1$). Show that the probability that on exactly k of these days more than l of the coins will each produce fewer than m heads can be approximated by

$$\binom{K}{k} q^k (1 - q)^{K-k},$$

where

$$q = \Phi\left(\frac{2h - 2l - 1}{2\sqrt{h}}\right), \quad h = L\Phi\left(\frac{2m - 1 - 2Mp}{2\sqrt{Mp(1-p)}}\right)$$

and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variate.

Would you expect this approximation to be accurate in the case $K = 7$, $k = 2$, $L = 500$, $l = 4$, $M = 100$, $m = 48$ and $p = 0.6$?

- 13 Let $F(x)$ be the cumulative distribution function of a random variable X , which satisfies $F(a) = 0$ and $F(b) = 1$, where $a > 0$. Let

$$G(y) = \frac{F(y)}{2 - F(y)}.$$

Show that $G(a) = 0$, $G(b) = 1$ and that $G'(y) \geq 0$. Show also that

$$\frac{1}{2} \leq \frac{2}{(2 - F(y))^2} \leq 2.$$

The random variable Y has cumulative distribution function $G(y)$. Show that

$$\frac{1}{2} E(X) \leq E(Y) \leq 2E(X),$$

and that

$$\text{Var}(Y) \leq 2\text{Var}(X) + \frac{7}{4}(E(X))^2.$$

- 14 A densely populated circular island is divided into N concentric regions R_1, R_2, \dots, R_N , such that the inner and outer radii of R_n are $n - 1$ km and n km, respectively. The average number of road accidents that occur in any one day in R_n is $2 - n/N$, independently of the number of accidents in any other region.

Each day an observer selects a region at random, with a probability that is proportional to the area of the region, and records the number of road accidents, X , that occur in it. Show that, in the long term, the average number of recorded accidents per day will be

$$2 - \frac{1}{6} \left(1 + \frac{1}{N}\right) \left(4 - \frac{1}{N}\right).$$

[Note: $\sum_{n=1}^N n^2 = \frac{1}{6}N(N+1)(2N+1)$.]

Show also that

$$P(X = k) = \frac{e^{-2}N^{-k-2}}{k!} \sum_{n=1}^N (2n-1)(2N-n)^k e^{n/N}.$$

Suppose now that $N = 3$ and that, on a particular day, two accidents were recorded. Show that the probability that R_2 had been selected is

$$\frac{48}{48 + 45e^{1/3} + 25e^{-1/3}}.$$