Section A: Pure Mathematics

1 It is given that $\sum\limits_{r=-1}^{n} r^2$ can be written in the form $pn^3 + qn^2 + rn + s$, where p, q, r and s are numbers. By setting $n = -1, 0, 1$ and 2, obtain four equations that must be satisfied by p, q, r and s and hence show that

$$\sum_{r=0}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\ .$$

Given that $\sum\limits_{r=-2}^{n} r^3$ can be written in the form $an^4 + bn^3 + cn^2 + dn + e$, show similarly that

$$\sum_{r=0}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2\ .$$

2 The first question on an examination paper is:

Solve for x the equation $\quad \dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}\ .$

where (in the question) a and b are given non-zero real numbers. One candidate writes $x = a + b$ as the solution. Show that there are no values of a and b for which this will give the correct answer.

The next question on the examination paper is:

Solve for x the equation $\quad \dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}\ .$

where (in the question) a, b and c are given non-zero numbers. The candidate uses the same technique, giving the answer as $x = a + b + c$. Show that the candidate's answer will be correct if and only if a, b and c satisfy at least one of the equations $a + b = 0$, $b + c = 0$ or $c + a = 0$.

3 **(i)** Show that $2\sin(\frac{1}{2}\theta) = \sin\theta$ if and only if $\sin(\frac{1}{2}\theta) = 0$.

(ii) Solve the equation $2\tan(\frac{1}{2}\theta) = \tan\theta$.

(iii) Show that $2\cos(\frac{1}{2}\theta) = \cos\theta$ if and only if $\theta = (4n+2)\pi \pm 2\phi$ where ϕ is defined by $\cos\phi = \frac{1}{2}(\sqrt{3} - 1)$, $0 \leqslant \phi \leqslant \frac{1}{2}\pi$, and n is any integer.

4 Solve the inequality

$$\frac{\sin\theta + 1}{\cos\theta} \leqslant 1$$

where $0 \leqslant \theta < 2\pi$ and $\cos\theta \neq 0$.

5 **(i)** In the binomial expansion of $(2x + 1/x^2)^6$ for $x \neq 0$, show that the term which is independent of x is 240.

Find the term which is independent of x in the binomial expansion of $(ax^3 + b/x^2)^{5n}$.

(ii) Let $\mathrm{f}(x) = (x^6 + 3x^5)^{1/2}$. By considering the expansion of $(1 + 3/x)^{1/2}$ show that the term which is independent of x in the expansion of $\mathrm{f}(x)$ in powers of $1/x$, for $|x| > 3$, is $27/16$.

Show that there is no term independent of x in the expansion of $\mathrm{f}(x)$ in powers of x, for $|x| < 3$.

6 Evaluate the following integrals, in the different cases that arise according to the value of the positive constant a:

(i) $$\int_0^1 \frac{1}{x^2 + (a+2)x + 2a}\,\mathrm{d}x\ ;$$

(ii) $$\int_1^2 \frac{1}{u^2 + au + a - 1}\,\mathrm{d}u\ .$$

7 Let k be an integer satisfying $0 \leqslant k \leqslant 9$. Show that $0 \leqslant 10k - k^2 \leqslant 25$ and that $k(k-1)(k+1)$ is divisible by 3.

For each 3-digit number N, where $N \geqslant 100$, let S be the sum of the hundreds digit, the square of the tens digit and the cube of the units digit. Find the numbers N such that $S = N$.

[Hint: write $N = 100a + 10b + c$ where a, b and c are the digits of N.]

8 A liquid of fixed volume V is made up of two chemicals A and B. A reaction takes place in which A converts to B. The volume of A at time t is xV and the volume of B at time t is yV where x and y depend on t and $x + y = 1$. The rate at which A converts into B is given by $kVxy$, where k is a positive constant. Show that if both x and y are strictly positive at the start, then at time t

$$y = \frac{D\mathrm{e}^{kt}}{1 + D\mathrm{e}^{kt}} \,,$$

where D is a constant.

Does A ever completely convert to B? Justify your answer.

Section B: Mechanics

9 A particle is projected with speed V at an angle θ above the horizontal. The particle passes through the point P which is a horizontal distance d and a vertical distance h from the point of projection. Show that

$$T^2 - 2kT + \frac{2kh}{d} + 1 = 0 \,,$$

where $T = \tan\theta$ and $k = \dfrac{V^2}{gd}$.

Show that, if $kd > h + \sqrt{h^2 + d^2}$, there are two distinct possible angles of projection.

Let these two angles be α and β. Show that $\alpha + \beta = \pi - \arctan(d/h)$.

10 $ABCD$ is a uniform rectangular lamina and X is a point on BC . The lengths of AD, AB and BX are p , q and r respectively. The triangle ABX is cut off the lamina. Let (a, b) be the position of the centre of gravity of the lamina, where the axes are such that the coordinates of A , D and C are $(0, 0)$, $(p, 0)$ and (p, q) respectively. Derive equations for a and b in terms of p , q and r .

When the resulting trapezium is freely suspended from the point A , the side AD is inclined at 45° below the horizontal. Show that $r = q - \sqrt{q^2 - 3pq + 3p^2}$. You should justify carefully the choice of sign in front of the square root.

11 A smooth plane is inclined at an angle α to the horizontal. A and B are two points a distance d apart on a line of greatest slope of the plane, with B higher than A. A particle is projected up the plane from A towards B with initial speed u, and simultaneously another particle is released from rest at B . Show that they collide after a time d/u .

The coefficient of restitution between the two particles is e and both particles have mass m . Show that the loss of kinetic energy in the collision is $\frac{1}{4}mu^2\big(1 - e^2\big)$.

Section C: Probability and Statistics

12 In a bag are n balls numbered 1, 2, . . . , n . When a ball is taken out of the bag, each ball is equally likely to be taken.

(i) A ball is taken out of the bag. The number on the ball is noted and the ball is replaced in the bag. The process is repeated once. Explain why the expected value of the product of the numbers on the two balls is

$$\frac{1}{n^2}\sum_{r=1}^{n}\sum_{s=1}^{n} rs$$

and simplify this expression.

(ii) A ball is taken out of the bag. The number on the ball is noted and the ball is *not* replaced in the bag. Another ball is taken out of the bag and the number on this ball is noted. Show that the expected value of the product of the two numbers is

$$\frac{(n+1)(3n+2)}{12} \, .$$

Note: $\displaystyle\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\displaystyle\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

13 If a football match ends in a draw, there may be a "penalty shoot-out". Initially the teams each take 5 shots at goal. If one team scores more times than the other, then that team wins. If the scores are level, the teams take shots alternately until one team scores and the other team does not score, both teams having taken the same number of shots. The team that scores wins.

Two teams, Team A and Team B, take part in a penalty shoot-out. Their probabilities of scoring when they take a single shot are p_A and p_B respectively. Explain why the probability α of neither side having won at the end of the initial 10-shot period is given by

$$\alpha = \sum_{i=0}^{5} \binom{5}{i}^2 (1-p_A)^i (1-p_B)^i \, p_A^{5-i} p_B^{5-i}.$$

Show that the expected number of shots taken is $10 + \dfrac{2\alpha}{\beta}$, where $\beta = p_A + p_B - 2p_A p_B$.

14 Jane goes out with any of her friends who call, except that she never goes out with more than two friends in a day. The number of her friends who call on a given day follows a Poisson distribution with parameter 2. Show that the average number of friends she sees in a day is $2 - 4\mathrm{e}^{-2}$.

Now Jane has a new friend who calls on any given day with probability p. Her old friends call as before, independently of the new friend. She never goes out with more than two friends in a day. Find the average number of friends she now sees in a day.