

Section A: Pure Mathematics

- 1 Given that $x + a > 0$ and $x + b > 0$, and that $b > a$, show that

$$\frac{d}{dx} \arcsin \left(\frac{x+a}{x+b} \right) = \frac{\sqrt{b-a}}{(x+b)\sqrt{a+b+2x}}$$

and find $\frac{d}{dx} \operatorname{arcosh} \left(\frac{x+b}{x+a} \right)$.

Hence, or otherwise, integrate, for $x > -1$,

(i) $\int \frac{1}{(x+1)\sqrt{x+3}} dx$,

(ii) $\int \frac{1}{(x+3)\sqrt{x+1}} dx$.

[You may use the results $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx} \operatorname{arcosh} x = \frac{1}{\sqrt{x^2-1}}$.]

- 2 Show that ${}^{2r}C_r = \frac{1 \times 3 \times \dots \times (2r-1)}{r!} \times 2^r$, for $r \geq 1$.

- (i) Give the first four terms of the binomial series for $(1-p)^{-\frac{1}{2}}$.

By choosing a suitable value for p in this series, or otherwise, show that

$$\sum_{r=0}^{\infty} \frac{{}^{2r}C_r}{8^r} = \sqrt{2}.$$

- (ii) Show that

$$\sum_{r=0}^{\infty} \frac{(2r+1) {}^{2r}C_r}{5^r} = (\sqrt{5})^3.$$

[Note: nC_r is an alternative notation for $\binom{n}{r}$ for $r \geq 1$, and ${}^0C_0 = 1$.]

- 3 If m is a positive integer, show that $(1+x)^m + (1-x)^m \neq 0$ for any real x .

The function f is defined by

$$f(x) = \frac{(1+x)^m - (1-x)^m}{(1+x)^m + (1-x)^m}.$$

Find and simplify an expression for $f'(x)$.

In the case $m = 5$, sketch the curves $y = f(x)$ and $y = \frac{1}{f(x)}$.

- 4 A curve is defined parametrically by

$$x = t^2, \quad y = t(1+t^2).$$

The tangent at the point with parameter t , where $t \neq 0$, meets the curve again at the point with parameter T , where $T \neq t$. Show that

$$T = \frac{1-t^2}{2t} \quad \text{and} \quad 3t^2 \neq 1.$$

Given a point P_0 on the curve, with parameter t_0 , a sequence of points P_0, P_1, P_2, \dots on the curve is constructed such that the tangent at P_i meets the curve again at P_{i+1} . If $t_0 = \tan \frac{7}{18}\pi$, show that $P_3 = P_0$ but $P_1 \neq P_0$. Find a second value of t_0 , with $t_0 > 0$, for which $P_3 = P_0$ but $P_1 \neq P_0$.

- 5 Find the coordinates of the turning point on the curve $y = x^2 - 2bx + c$. Sketch the curve in the case that the equation $x^2 - 2bx + c = 0$ has two distinct real roots. Use your sketch to determine necessary and sufficient conditions on b and c for the equation $x^2 - 2bx + c = 0$ to have two distinct real roots. Determine necessary and sufficient conditions on b and c for this equation to have two distinct positive roots.

Find the coordinates of the turning points on the curve $y = x^3 - 3b^2x + c$ (with $b > 0$) and hence determine necessary and sufficient conditions on b and c for the equation $x^3 - 3b^2x + c = 0$ to have three distinct real roots. Determine necessary and sufficient conditions on a, b and c for the equation $(x-a)^3 - 3b^2(x-a) + c = 0$ to have three distinct positive roots.

Show that the equation $2x^3 - 9x^2 + 7x - 1 = 0$ has three distinct positive roots.

6 Show that

$$2 \sin \frac{1}{2} \theta \cos r \theta = \sin \left(r + \frac{1}{2} \right) \theta - \sin \left(r - \frac{1}{2} \right) \theta .$$

Hence, or otherwise, find all solutions of the equation

$$\cos a \theta + \cos(a+1)\theta + \cdots + \cos(b-2)\theta + \cos(b-1)\theta = 0 ,$$

where a and b are positive integers with $a < b - 1$.

7 In the x - y plane, the point A has coordinates $(a, 0)$ and the point B has coordinates $(0, b)$, where a and b are positive. The point P , which is distinct from A and B , has coordinates (s, t) . X and Y are the feet of the perpendiculars from P to the x -axis and y -axis respectively, and N is the foot of the perpendicular from P to the line AB . Show that the coordinates (x, y) of N are given by

$$x = \frac{ab^2 - a(bt - as)}{a^2 + b^2} , \quad y = \frac{a^2b + b(bt - as)}{a^2 + b^2} .$$

Show that, if $\left(\frac{t-b}{s} \right) \left(\frac{t}{s-a} \right) = -1$, then N lies on the line XY .

Give a geometrical interpretation of this result.

8 (i) Show that the gradient at a point (x, y) on the curve

$$(y + 2x)^3 (y - 4x) = c ,$$

where c is a constant, is given by

$$\frac{dy}{dx} = \frac{16x - y}{2y - 5x} .$$

(ii) By considering the derivative with respect to x of $(y + ax)^n (y + bx)$, or otherwise, find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{10x - 4y}{3x - y} .$$

Section B: Mechanics

- 9 A particle P of mass m is constrained to move on a vertical circle of smooth wire with centre O and of radius a . L is the lowest point of the circle and H the highest and $\angle LOP = \theta$. The particle is attached to H by an elastic string of natural length a and modulus of elasticity αmg , where $\alpha > 1$. Show that, if $\alpha > 2$, there is an equilibrium position with $0 < \theta < \pi$.

Given that $\alpha = 2 + \sqrt{2}$, and that $\theta = \frac{1}{2}\pi + \phi$, show that

$$\ddot{\phi} \approx -\frac{g(\sqrt{2} + 1)}{2a} \phi$$

when ϕ is small.

For this value of α , explain briefly what happens to the particle if it is given a small displacement when $\theta = \frac{1}{2}\pi$.

- 10 A particle moves along the x -axis in such a way that its acceleration is $kx\dot{x}$ where k is a positive constant. When $t = 0$, $x = d$ (where $d > 0$) and $\dot{x} = U$.

- (i) Find x as a function of t in the case $U = kd^2$ and show that x tends to infinity as t tends to $\frac{\pi}{2dk}$.
- (ii) If $U < 0$, find x as a function of t and show that it tends to a limit, which you should state in terms of d and U , as t tends to infinity.

- 11 Point B is a distance d due south of point A on a horizontal plane. Particle P is at rest at B at $t = 0$, when it begins to move with constant acceleration a in a straight line with fixed bearing β . Particle Q is projected from point A at $t = 0$ and moves in a straight line with constant speed v . Show that if the direction of projection of Q can be chosen so that Q strikes P , then

$$v^2 \geq ad(1 - \cos \beta) .$$

Show further that if $v^2 > ad(1 - \cos \beta)$ then the direction of projection of Q can be chosen so that Q strikes P before P has moved a distance d .

Section C: Probability and Statistics

- 12** Brief interruptions to my work occur on average every ten minutes and the number of interruptions in any given time period has a Poisson distribution. Given that an interruption has just occurred, find the probability that I will have less than t minutes to work before the next interruption. If the random variable T is the time I have to work before the next interruption, find the probability density function of T .

I need an uninterrupted half hour to finish an important paper. Show that the expected number of interruptions before my first uninterrupted period of half an hour or more is $e^3 - 1$. Find also the expected length of time between interruptions that are less than half an hour apart. Hence write down the expected wait before my first uninterrupted period of half an hour or more.

- 13** In a rabbit warren, underground chambers A, B, C and D are at the vertices of a square, and burrows join A to B , B to C , C to D and D to A . Each of the chambers also has a tunnel to the surface. A rabbit finding itself in any chamber runs along one of the two burrows to a neighbouring chamber, or leaves the burrow through the tunnel to the surface. Each of these three possibilities is equally likely.

Let p_A, p_B, p_C and p_D be the probabilities of a rabbit leaving the burrow through the tunnel from chamber A, B, C or D , given that it is currently in chamber A, B, C or D , respectively.

(i) Explain why $p_A = \frac{1}{3} + \frac{1}{3}p_B + \frac{1}{3}p_D$.

(ii) Determine p_A .

- (iii) Find the probability that a rabbit which starts in chamber A does not visit chamber C , given that it eventually leaves the burrow through the tunnel in chamber A .

- 14 Write down the probability generating function for the score on a standard, fair six-faced die whose faces are labelled 1, 2, 3, 4, 5, 6. Hence show that the probability generating function for the sum of the scores on two standard, fair six-faced dice, rolled independently, can be written as

$$\frac{1}{36}t^2(1+t)^2(1-t+t^2)^2(1+t+t^2)^2.$$

Write down, in factorised form, the probability generating functions for the scores on two fair six-faced dice whose faces are labelled with the numbers 1, 2, 2, 3, 3, 4 and 1, 3, 4, 5, 6, 8, and hence show that when these dice are rolled independently, the probability of any given sum of the scores is the same as for the two standard fair six-faced dice.

Standard, fair four-faced dice are tetrahedra whose faces are labelled 1, 2, 3, 4, the score being taken from the face which is not visible after throwing, and each score being equally likely. Find all the ways in which two fair four-faced dice can have their faces labelled with positive integers if the probability of any given sum of the scores is to be the same as for the two standard fair four-faced dice.