Section A: Pure Mathematics

1 Show that

$$\int_0^a \frac{\sinh x}{2\cosh^2 x - 1}\,\mathrm{d}x = \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}\cosh a - 1}{\sqrt{2}\cosh a + 1}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$

and find

$$\int_0^a \frac{\cosh x}{1 + 2\sinh^2 x}\,\mathrm{d}x\,.$$

Hence show that

$$\int_0^\infty \frac{\cosh x - \sinh x}{1 + 2\sinh^2 x}\,\mathrm{d}x = \frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\,.$$

By substituting $u = \mathrm{e}^x$ in this result, or otherwise, find

$$\int_1^\infty \frac{1}{1+u^4}\,\mathrm{d}u\,.$$

2 The equation of a curve is $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = x - 4 - \frac{16\,(2x+1)^2}{x^2\,(x-4)}\,.$$

(i) Write down the equations of the vertical and oblique asymptotes to the curve and show that the oblique asymptote is a tangent to the curve.

(ii) Show that the equation $\mathrm{f}(x) = 0$ has a double root.

(iii) Sketch the curve.

3 Given that $\mathrm{f}''(x) > 0$ when $a \leqslant x \leqslant b$, explain with the aid of a sketch why

$$(b-a)\,\mathrm{f}\Big(\frac{a+b}{2}\Big) < \int_a^b \mathrm{f}(x)\,\mathrm{d}x < (b-a)\,\frac{\mathrm{f}(a)+\mathrm{f}(b)}{2}\ .$$

By choosing suitable a, b and $\mathrm{f}(x)$, show that

$$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{n^2} + \frac{1}{(n-1)^2}\right),$$

where n is an integer greater than 1.

Deduce that

$$4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right).$$

Show that

$$\frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots\right) < \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

and hence show that

$$\frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n^2} < \frac{7}{4}\ .$$

4 The triangle OAB is isosceles, with $OA = OB$ and angle $AOB = 2\alpha$ where $0 < \alpha < \frac{\pi}{2}$. The semi-circle C_0 has its centre at the midpoint of the base AB of the triangle, and the sides OA and OB of the triangle are both tangent to the semi-circle. $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \ldots$ are circles such that C_n is tangent to C_{n-1} and to sides OA and OB of the triangle.

Let r_n be the radius of C_n. Show that

$$\frac{r_{n+1}}{r_n} = \frac{1-\sin\alpha}{1+\sin\alpha}\ .$$

Let S be the total area of the semi-circle C_0 and the circles C_1, C_2, C_3, $\ldots$. Show that

$$S = \frac{1+\sin^2\alpha}{4\sin\alpha}\,\pi r_0^2\ .$$

Show that there are values of α for which S is more than four fifths of the area of triangle OAB.

5 Show that if $\cos(x-\alpha)=\cos\beta$ then either $\tan x=\tan(\alpha+\beta)$ or $\tan x=\tan(\alpha-\beta)$. By choosing suitable values of x, α and β, give an example to show that if $\tan x=\tan(\alpha+\beta)$, then $\cos(x-\alpha)$ need not equal $\cos\beta$.

Let ω be the acute angle such that $\tan\omega=\frac{4}{3}$.

(i) For $0\leqslant x\leqslant 2\pi$, solve the equation

$$\cos x-7\sin x=5$$

giving both solutions in terms of ω.

(ii) For $0\leqslant x\leqslant 2\pi$, solve the equation

$$2\cos x+11\sin x=10$$

showing that one solution is twice the other and giving both in terms of ω.

6 Given a sequence $w_0, w_1, w_2, \ldots$, the sequence $F_1, F_2, \ldots$ is defined by

$$F_n=w_n^2+w_{n-1}^2-4w_nw_{n-1}\,.$$

Show that $F_n-F_{n-1}=(w_n-w_{n-2})\,(w_n+w_{n-2}-4w_{n-1})$ for $n\geqslant 2$.

(i) The sequence $u_0, u_1, u_2, \ldots$ has $u_0=1$, and $u_1=2$ and satisfies

$$u_n=4u_{n-1}-u_{n-2}\quad(n\geqslant 2)\,.$$

Prove that $u_n^2+u_{n-1}^2=4u_nu_{n-1}-3$ for $n\geqslant 1$.

(ii) A sequence $v_0, v_1, v_2, \ldots$ has $v_0=1$ and satisfies

$$v_n^2+v_{n-1}^2=4v_nv_{n-1}-3\quad(n\geqslant 1). \qquad (*)$$

(a) Find v_1 and prove that, for each $n\geqslant 2$, either $v_n=4v_{n-1}-v_{n-2}$ or $v_n=v_{n-2}$.

(b) Show that the sequence, with period 2, defined by

$$v_n=\begin{cases}1 & \text{for } n \text{ even}\\ 2 & \text{for } n \text{ odd}\end{cases}$$

satisfies $(*)$.

(c) Find a sequence v_n with period 4 which has $v_0=1$, and satisfies $(*)$.

7 For $n = 1, 2, 3, \ldots$, let

$$I_n = \int_0^1 \frac{t^{n-1}}{(t+1)^n}\,\mathrm{d}t\,.$$

By considering the greatest value taken by $\dfrac{t}{t+1}$ for $0 \leqslant t \leqslant 1$ show that $I_{n+1} < \frac{1}{2}I_n$.

Show also that $I_{n+1} = -\dfrac{1}{n\,2^n} + I_n$.

Deduce that $I_n < \dfrac{1}{n\,2^{n-1}}$.

Prove that

$$\ln 2 = \sum_{r=1}^{n} \frac{1}{r\,2^r} + I_{n+1}$$

and hence show that $\frac{2}{3} < \ln 2 < \frac{17}{24}$.

8 Show that if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)y + \frac{\mathrm{g}(x)}{y}$$

then the substitution $u = y^2$ gives a linear differential equation for $u(x)$.

Hence or otherwise solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} - \frac{1}{y}\,.$$

Determine the solution curves of this equation which pass through $(1\,,1)$, $(2\,,2)$ and $(4\,,4)$ and sketch graphs of all three curves on the same axes.

Section B: Mechanics

9 A circular hoop of radius a is free to rotate about a fixed horizontal axis passing through a point P on its circumference. The plane of the hoop is perpendicular to this axis. The hoop hangs in equilibrium with its centre, O, vertically below P. The point A on the hoop is vertically below O, so that POA is a diameter of the hoop.

A mouse M runs at constant speed u round the rough inner surface of the lower part of the hoop. Show that the mouse can choose its speed so that the hoop remains in equilibrium with diameter POA vertical.

Describe what happens to the hoop when the mouse passes the point at which angle $AOM = 2\arctan\mu$, where μ is the coefficient of friction between mouse and hoop.

10 A particle P of mass m is attached to points A and B, where A is a distance $9a$ vertically above B, by elastic strings, each of which has modulus of elasticity $6mg$. The string AP has natural length $6a$ and the string BP has natural length $2a$. Let x be the distance AP.

The system is released from rest with P on the vertical line AB and $x = 6a$. Show that the acceleration $\ddot{x}$ of P is $\dfrac{4g}{a}(7a - x)$ for $6a < x < 7a$ and $\dfrac{g}{a}(7a - x)$ for $7a < x < 9a$.

Find the time taken for the particle to reach B.

11 Particles P, of mass 2, and Q, of mass 1, move along a line. Their distances from a fixed point are x_1 and x_2, respectively where $x_2 > x_1$. Each particle is subject to a repulsive force from the other of magnitude $\dfrac{2}{z^3}$, where $z = x_2 - x_1$.

Initially, $x_1 = 0$, $x_2 = 1$, Q is at rest and P moves towards Q with speed 1. Show that z obeys the equation $\dfrac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \dfrac{3}{z^3}$.

By first writing $\dfrac{\mathrm{d}^2 z}{\mathrm{d}t^2} = v\dfrac{\mathrm{d}v}{\mathrm{d}z}$, where $v = \dfrac{\mathrm{d}z}{\mathrm{d}t}$, show that $z = \sqrt{4t^2 - 2t + 1}$.

By considering the equation satisfied by $2x_1 + x_2$, find x_1 and x_2 in terms of t.

Section C: Probability and Statistics

12 A team of m players, numbered from 1 to m, puts on a set of a m shirts, similarly numbered from 1 to m. The players change in a hurry, so that the shirts are assigned to them randomly, one to each player.

Let C_i be the random variable that takes the value 1 if player i is wearing shirt i, and 0 otherwise. Show that $\mathrm{E}\,(C_1) = \frac{1}{m}$ and find $\mathrm{Var}\ (C_1)$ and $\mathrm{Cov}\,(C_1\,,\ C_2)\,$.

Let $N = C_1 + C_2 + \cdots + C_m$ be the random variable whose value is the number of players who are wearing the correct shirt. Show that $\mathrm{E}\,(N) = \mathrm{Var}\ (N) = 1\,$.

Explain why a Normal approximation to N is not likely to be appropriate for any m, but that a Poisson approximation might be reasonable.

In the case $m = 4$, find, by listing equally likely possibilities or otherwise, the probability that no player is wearing the correct shirt and verify that an appropriate Poisson approximation to N gives this probability with a relative error of about 2%. [Use $\mathrm{e} \approx 2\frac{72}{100}$.]

13 A men's endurance competition has an unlimited number of rounds. In each round, a competitor has, independently, a probability p of making it through the round; otherwise, he fails the round. Once a competitor fails a round, he drops out of the competition; before he drops out, he takes part in every round. The grand prize is awarded to any competitor who makes it through a round which all the other remaining competitors fail; if all the remaining competitors fail at the same round the grand prize is not awarded.

If the competition begins with three competitors, find the probability that:

(i) all three drop out in the same round;

(ii) two of them drop out in round r (with $r \geqslant 2$) and the third in an earlier round;

(iii) the grand prize is awarded.

14 *In this question, $\Phi(z)$ is the cumulative distribution function of a standard normal random variable.*

A random variable is known to have a Normal distribution with mean μ and standard deviation either σ_0 or σ_1, where $\sigma_0 < \sigma_1$. The mean, $\overline{X}$, of a random sample of n values of X is to be used to test the hypothesis $\mathrm{H}_0 : \sigma = \sigma_0$ against the alternative $\mathrm{H}_1 : \sigma = \sigma_1$.

Explain carefully why it is appropriate to use a two sided test of the form: accept H_0 if $\mu - c < \overline{X} < \mu + c$, otherwise accept H_1.

Given that the probability of accepting H_1 when H_0 is true is α, determine c in terms of n, σ_0 and z_α, where z_α is defined by $\Phi(z_\alpha) = 1 - \frac{1}{2}\alpha$.

The probability of accepting H_0 when H_1 is true is denoted by β. Show that β is independent of n.

Given that $\Phi(1.960) \approx 0.975$ and that $\Phi(0.063) \approx 0.525$, determine, approximately, the minimum value of $\dfrac{\sigma_1}{\sigma_0}$ if α and β are both to be less than 0.05.