Section A: Pure Mathematics

1 A positive integer with $2n$ digits (the first of which must not be 0) is called a *balanced number* if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

(i) Show that seventy 4-digit balanced numbers can be made using the digits $0, 1, 2, 3$ and 4.

(ii) Show that $\frac{1}{6}k\,(k+1)\,(4k+5)$ 4-digit balanced numbers can be made using the digits 0 to k.

You may use the identity $\displaystyle\sum_{r=0}^{n} r^2 \equiv \frac{1}{6}n\,(n+1)\,(2n+1)$ *.*

2 **(i)** Given that $A = \arctan\frac{1}{2}$ and that $B = \arctan\frac{1}{3}$ (where A and B are acute) show, by considering $\tan(A+B)$, that $A + B = \frac{1}{4}\pi$.

The non-zero integers p and q satisfy

$$\arctan\frac{1}{p} + \arctan\frac{1}{q} = \frac{\pi}{4}\,.$$

Show that $(p-1)\,(q-1) = 2$ and hence determine p and q.

(ii) Let r, s and t be positive integers such that the highest common factor of s and t is 1. Show that, if

$$\arctan\frac{1}{r} + \arctan\frac{s}{s+t} = \frac{\pi}{4}\,,$$

then there are only two possible values for t, and give r in terms of s in each case.

3 Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$. Hence or otherwise evaluate

$$\int_0^{\frac{1}{2}\pi} \cos^4\theta \,\mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^4\theta \,\mathrm{d}\theta\,.$$

Evaluate also

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \,\mathrm{d}\theta \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \sin^6\theta \,\mathrm{d}\theta\,.$$

4 Show that $x^3 - 3xbc + b^3 + c^3$ can be written in the form $(x + b + c)\,\mathrm{Q}(x)$, where $\mathrm{Q}(x)$ is a quadratic expression. Show that $2\mathrm{Q}(x)$ can be written as the sum of three expressions, each of which is a perfect square.

It is given that the equations $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k. The coefficients a, b and c are real, a and b are both non-zero, and $ac \neq b^2$. Show that

$$\left(ac - b^2\right) k = bc - a^2$$

and determine a similar expression involving k^2. Hence show that

$$\left(ac - b^2\right)\left(ab - c^2\right) = \left(bc - a^2\right)^2$$

and that $a^3 - 3abc + b^3 + c^3 = 0$. Deduce that either $k = 1$ or the two equations are identical.

5 *Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.*

(i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$.

(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

6 **(i)** Given that $x^2 - y^2 = (x - y)^3$ and that $x - y = d$ (where $d \neq 0$), express each of x and y in terms of d. Hence find a pair of integers m and n satisfying $m - n = (\sqrt{m} - \sqrt{n})^3$ where $m > n > 100$.

(ii) Given that $x^3 - y^3 = (x - y)^4$ and that $x - y = d$ (where $d \neq 0$), show that $3xy = d^3 - d^2$. Hence show that

$$2x = d \pm d\sqrt{\frac{4d - 1}{3}}$$

and determine a pair of distinct positive integers m and n such that $m^3 - n^3 = (m - n)^4$.

7 **(i)** The line L_1 has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$.

The line L_2 has vector equation $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

Show that the distance D between a point on L_1 and a point on L_2 can be expressed in the form

$$D^2 = (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36\,.$$

Hence determine the minimum distance between these two lines and find the coordinates of the points on the two lines that are the minimum distance apart.

(ii) The line L_3 has vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The line L_4 has vector equation $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} + \beta \begin{pmatrix} 4k \\ 1-k \\ -3k \end{pmatrix}$.

Determine the minimum distance between these two lines, explaining geometrically the two different cases that arise according to the value of k.

8 A curve is given by the equation

$$y = ax^3 - 6ax^2 + (12a + 12)\,x - (8a + 16)\ , \qquad (*)$$

where a is a real number. Show that this curve touches the curve with equation

$$y = x^3 \qquad (**)$$

at $(2\,,\,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 2$.

(ii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 1$.

(iii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = -2$.

Section B: Mechanics

9 A particle of weight W is placed on a rough plane inclined at an angle of θ to the horizontal. The coefficient of friction between the particle and the plane is μ. A horizontal force X acting on the particle is just sufficient to prevent the particle from sliding down the plane; when a horizontal force kX acts on the particle, the particle is about to slide up the plane. Both horizontal forces act in the vertical plane containing the line of greatest slope.

Prove that

$$(k-1)\left(1+\mu^2\right)\sin\theta\cos\theta = \mu\,(k+1)$$

and hence that $k \geqslant \dfrac{(1+\mu)^2}{(1-\mu)^2}$.

10 The Norman army is advancing with constant speed u towards the Saxon army, which is at rest. When the armies are d apart, a Saxon horseman rides from the Saxon army directly towards the Norman army at constant speed x. Simultaneously a Norman horseman rides from the Norman army directly towards the Saxon army at constant speed y, where $y > u$. The horsemen ride their horses so that $y - 2x < u < 2y - x$.

When each horseman reaches the opposing army, he immediately rides straight back to his own army without changing his speed. Represent this information on a displacement-time graph, and show that the two horsemen pass each other at distances

$$\frac{xd}{x+y} \quad \text{and} \quad \frac{xd(2y-x-u)}{(u+x)(x+y)}$$

from the Saxon army.

Explain briefly what will happen in the cases (i) $u > 2y - x$ and (ii) $u < y - 2x$.

11 A smooth, straight, narrow tube of length L is fixed at an angle of 30° to the horizontal. A particle is fired up the tube, from the lower end, with initial velocity u. When the particle reaches the upper end of the tube, it continues its motion until it returns to the same level as the lower end of the tube, having travelled a horizontal distance D after leaving the tube. Show that D satisfies the equation

$$4gD^2 - 2\sqrt{3}\left(u^2 - Lg\right)D - 3L\left(u^2 - gL\right) = 0$$

and hence that

$$\frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\,(u^2 - gL)}.$$

The final horizontal displacement of the particle from the lower end of the tube is R. Show that $\dfrac{\mathrm{d}R}{\mathrm{d}L} = 0$ when $2D = L\sqrt{3}$, and determine, in terms of u and g, the corresponding value of R.

Section C: Probability and Statistics

12 **(i)** A bag contains N sweets (where $N \geqslant 2$), of which a are red. Two sweets are drawn from the bag without replacement. Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

(ii) There are two bags, each containing N sweets (where $N \geqslant 2$). The first bag contains a red sweets, and the second bag contains b red sweets. There is also a biased coin, showing Heads with probability p and Tails with probability q, where $p + q = 1$.

The coin is tossed. If it shows Heads then a sweet is chosen from the first bag and transferred to the second bag; if it shows Tails then a sweet is chosen from the second bag and transferred to the first bag. The coin is then tossed a second time: if it shows Heads then a sweet is chosen from the first bag, and if it shows Tails then a sweet is chosen from the second bag.

Show that the probability that the first sweet is red is equal to the probability that the second sweet is red.

13 A bag contains eleven small discs, which are identical except that six of the discs are blank and five of the discs are numbered, using the numbers $1, 2, 3, 4$ and 5. The bag is shaken, and four discs are taken one at a time without replacement.

Calculate the probability that:

(i) all four discs taken are numbered;

(ii) all four discs taken are numbered, given that the disc numbered "3" is taken first;

(iii) exactly two numbered discs are taken, given that the disc numbered "3" is taken first;

(iv) exactly two numbered discs are taken, given that the disc numbered "3" is taken;

(v) exactly two numbered discs are taken, given that a numbered disc is taken first;

(vi) exactly two numbered discs are taken, given that a numbered disc is taken.

14 The discrete random variable X has a Poisson distribution with mean λ.

(i) Sketch the graph $y = (x+1)\,\mathrm{e}^{-x}$, stating the coordinates of the turning point and the points of intersection with the axes.

It is known that $\mathrm{P}(X \geqslant 2) = 1 - p$, where p is a given number in the range $0 < p < 1$. Show that this information determines a unique value (which you should not attempt to find) of λ.

(ii) It is known (instead) that $\mathrm{P}\,(X = 1) = q$, where q is a given number in the range $0 < q < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of q (which you should also find).

(iii) It is known (instead) that $\mathrm{P}\,(X = 1 \,|\, X \leqslant 2) = r$, where r is a given number in the range $0 < r < 1$. Show that this information determines a unique value of λ (which you should find) for exactly one value of r (which you should also find).