Section A: Pure Mathematics

1 What does it mean to say that a number x is *irrational*?

Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.

B: If $p+q$ is irrational, then at least one of p and q is irrational.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p+q$ is irrational.

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi+\mathrm{e}$, $\pi-\mathrm{e}$, $\pi^2-\mathrm{e}^2$, $\pi^2+\mathrm{e}^2$ is rational.

2 The variables t and x are related by $t = x + \sqrt{x^2+2bx+c}$, where b and c are constants and $b^2 < c$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t-x}{t+b}\,,$$

and hence integrate $\dfrac{1}{\sqrt{x^2+2bx+c}}$.

Verify by direct integration that your result holds also in the case $b^2 = c$ if $x+b>0$ but that your result does not hold in the case $b^2=c$ if $x+b<0$.

3 Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad\,. \qquad (*)$$

(i) If $x \geqslant y$, use $(*)$ to show that $x^2+y^2 \geqslant 2xy$.

If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2+xy \geqslant xz+yz$ and deduce that $x^2+y^2+z^2 \geqslant xy+yz+zx$.

Prove that the inequality $x^2+y^2+z^2 \geqslant xy+yz+zx$ holds for all x, y and z.

(ii) Show similarly that the inequality

$$\frac{s}{t}+\frac{t}{r}+\frac{r}{s}+\frac{t}{s}+\frac{r}{t}+\frac{s}{r} \geqslant 6$$

holds for all positive r, s and t.

4 A function $\mathrm{f}(x)$ is said to be *convex* in the interval $a < x < b$ if $\mathrm{f}''(x) \geqslant 0$ for all x in this interval.

(i) Sketch on the same axes the graphs of $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$ in the interval $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ is defined for $0 < x < 2\pi$ by

$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ is convex.

(ii) The function $\mathrm{g}(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ and $0 < \alpha < \frac{1}{4}\pi$, show that $\mathrm{g}(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $\mathrm{g}(x)$ is convex.

5 The polynomial $\mathrm{p}(x)$ is given by

$$\mathrm{p}(x) = x^n + \sum_{r=0}^{n-1} a_r x^r \,,$$

where $a_0, a_1, \ldots, a_{n-1}$ are fixed real numbers and $n \geqslant 1$. Let M be the greatest value of $\left|\mathrm{p}(x)\right|$ for $|x| \leqslant 1$. Then *Chebyshev's theorem* states that $M \geqslant 2^{1-n}$.

(i) Prove Chebyshev's theorem in the case $n = 1$ and verify that Chebyshev's theorem holds in the following cases:

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}\,;$

(b) $\mathrm{p}(x) = x^3 - x\,.$

(ii) Use Chebyshev's theorem to show that the curve $y = 64x^5 + 25x^4 - 66x^3 - 24x^2 + 3x + 1$ has at least one turning point in the interval $-1 \leqslant x \leqslant 1$.

6 The function f is defined by

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

and the function g is the inverse function to f, so that $\mathrm{g}(\mathrm{f}(x)) = x$. Sketch $\mathrm{f}(x)$ and $\mathrm{g}(x)$ on the same axes.

Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^{k} \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}}+1)}\,,$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}}+1}$, and explain this result by means of a diagram.

7 The point P has coordinates (x, y) with respect to the origin O. By writing $x = r\cos\theta$ and $y = r\sin\theta$, or otherwise, show that, if the line OP is rotated by 60° clockwise about O, the new y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$. What is the new y-coordinate in the case of an anti-clockwise rotation by 60° ?

An equilateral triangle OBC has vertices at O, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, respectively. The point P has coordinates (x, y). The perpendicular distance from P to the line through C and O is h_1; the perpendicular distance from P to the line through O and B is h_2; and the perpendicular distance from P to the line through B and C is h_3.

Show that $h_1 = \frac{1}{2}\big|y - \sqrt{3}\,x\big|$ and find expressions for h_2 and h_3.

Show that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or in the triangle OBC.

8 **(i)** The gradient y' of a curve at a point (x, y) satisfies

$$(y')^2 - xy' + y = 0\,. \qquad (*)$$

By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.

Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.

(ii) The gradient y' of a curve at a point (x, y) satisfies

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0\,.$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

Section B: Mechanics

9 Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ. Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$. Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.

Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

10 On the (flat) planet Zog, the acceleration due to gravity is g up to height h above the surface and g' at greater heights. A particle is projected from the surface at speed V and at an angle α to the surface, where $V^2 \sin^2 \alpha > 2gh$. Sketch, on the same axes, the trajectories in the cases $g' = g$ and $g' < g$.

Show that the particle lands a distance d from the point of projection given by

$$d = \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) V \sin 2\alpha \,,$$

where $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2\alpha}$.

11 A straight uniform rod has mass m. Its ends P_1 and P_2 are attached to small light rings that are constrained to move on a rough rigid circular wire with centre O fixed in a vertical plane, and the angle P_1OP_2 is a right angle. The rod rests with P_1 lower than P_2, and with both ends lower than O. The coefficient of friction between each of the rings and the wire is μ. Given that the rod is in limiting equilibrium (i.e. on the point of slipping at both ends), show that

$$\tan \alpha = \frac{1 - 2\mu - \mu^2}{1 + 2\mu - \mu^2} \,,$$

where α is the angle between P_1O and the vertical ($0 < \alpha < 45°$).

Let θ be the acute angle between the rod and the horizontal. Show that $\theta = 2\lambda$, where λ is defined by $\tan \lambda = \mu$ and $0 < \lambda < 22.5°$.

Section C: Probability and Statistics

12 In this question, you may use without proof the results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)\,.$$

The independent random variables X_1 and X_2 each take values $1, 2, \ldots, N$, each value being equally likely. The random variable X is defined by

$$X = \begin{cases} X_1 & \text{if } X_1 \geqslant X_2 \\ X_2 & \text{if } X_2 \geqslant X_1\ . \end{cases}$$

(i) Show that $\mathrm{P}(X = r) = \dfrac{2r-1}{N^2}$ for $r = 1, 2, \ldots, N$.

(ii) Find an expression for the expectation, μ, of X and show that $\mu = 67.165$ in the case $N = 100$.

(iii) The median, m, of X is defined to be the integer such that $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$. Find an expression for m in terms of N and give an explicit value for m in the case $N = 100$.

(iv) Show that when N is very large,

$$\frac{\mu}{m} \approx \frac{2\sqrt{2}}{3}\,.$$

13 Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.

(ii) Find the probability that exactly two husbands sit next to their wives.

(iii) Find the probability that no husband sits next to his wife.