Section A: Pure Mathematics

1 A sequence of points (x_1, y_1), (x_2, y_2), ... in the cartesian plane is generated by first choosing (x_1, y_1) then applying the rule, for $n = 1, 2, \ldots$,

$$(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + a,\ 2x_n y_n + b + 2)\,,$$

where a and b are given real constants.

(i) In the case $a = 1$ and $b = -1$, find the values of (x_1, y_1) for which the sequence is constant.

(ii) Given that $(x_1, y_1) = (-1, 1)$, find the values of a and b for which the sequence has period 2.

2 Let a_n be the coefficient of x^n in the series expansion, in ascending powers of x, of

$$\frac{1+x}{(1-x)^2(1+x^2)}\,,$$

where $|x| < 1$. Show, using partial fractions, that either $a_n = n + 1$ or $a_n = n + 2$ according to the value of n.

Hence find a decimal approximation, to nine significant figures, for the fraction $\dfrac{11\,000}{8181}$.

[You are not required to justify the accuracy of your approximation.]

3 **(i)** Find the coordinates of the turning points of the curve $y = 27x^3 - 27x^2 + 4$. Sketch the curve and deduce that $x^2(1-x) \leqslant 4/27$ for all $x \geqslant 0$.

Given that each of the numbers a, b and c lies between 0 and 1, prove by contradiction that at least one of the numbers $bc(1-a)$, $ca(1-b)$ and $ab(1-c)$ is less than or equal to $4/27$.

(ii) Given that each of the numbers p and q lies between 0 and 1, prove that at least one of the numbers $p(1-q)$ and $q(1-p)$ is less than or equal to $1/4$.

4 A curve is given by

$$x^2 + y^2 + 2axy = 1,$$

where a is a constant satisfying $0 < a < 1$. Show that the gradient of the curve at the point P with coordinates (x, y) is

$$-\frac{x + ay}{ax + y},$$

provided $ax + y \neq 0$. Show that θ, the acute angle between OP and the normal to the curve at P, satisfies

$$\tan\theta = a|y^2 - x^2| \, .$$

Show further that, if $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0$ at P, then:

(i) $a(x^2 + y^2) + 2xy = 0$;

(ii) $(1 + a)(x^2 + y^2 + 2xy) = 1$;

(iii) $\tan\theta = \dfrac{a}{\sqrt{1 - a^2}}$.

5 Evaluate the integrals

$$\int_0^{\frac{1}{2}\pi} \frac{\sin 2x}{1 + \sin^2 x} \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{1}{2}\pi} \frac{\sin x}{1 + \sin^2 x} \mathrm{d}x \, .$$

Show, using the binomial expansion, that $(1 + \sqrt{2}\,)^5 < 99$. Show also that $\sqrt{2} > 1.4$. Deduce that $2^{\sqrt{2}} > 1 + \sqrt{2}$. Use this result to determine which of the above integrals is greater.

6 A curve has the equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \cos\left(2x + \frac{\pi}{3}\right) + \sin\left(\frac{3x}{2} - \frac{\pi}{4}\right).$$

(i) Find the period of $\mathrm{f}(x)$.

(ii) Determine all values of x in the interval $-\pi \leqslant x \leqslant \pi$ for which $\mathrm{f}(x) = 0$. Find a value of x in this interval at which the curve touches the x-axis without crossing it.

(iii) Find the value or values of x in the interval $0 \leqslant x \leqslant 2\pi$ for which $\mathrm{f}(x) = 2$.

7 **(i)** By writing $y = u(1+x^2)^{\frac{1}{2}}$, where u is a function of x, find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = xy + \frac{x}{1+x^2}$$

for which $y = 1$ when $x = 0$.

(ii) Find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 y + \frac{x^2}{1+x^3}$$

for which $y = 1$ when $x = 0$.

(iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^{n-1} y + \frac{x^{n-1}}{1+x^n}$$

for which $y = 1$ when $x = 0$, where n is an integer greater than 1.

8 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points A, B and O are not collinear. The point P lies on AB between A and B such that

$$AP : PB = (1-\lambda) : \lambda\,.$$

Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. Given that OP bisects $\angle AOB$, determine λ in terms of a and b, where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$.

The point Q also lies on AB between A and B, and is such that $AP = BQ$. Prove that

$$OQ^2 - OP^2 = (b-a)^2\,.$$

Section B: Mechanics

9 In this question, use $g = 10\,\text{m}\,\text{s}^{-2}$.

In cricket, a fast bowler projects a ball at $40\,\text{m}\,\text{s}^{-1}$ from a point h m above the ground, which is horizontal, and at an angle α above the horizontal. The trajectory is such that the ball will strike the stumps at ground level a horizontal distance of 20 m from the point of projection.

(i) Determine, in terms of h, the two possible values of $\tan\alpha$.

Explain which of these two values is the more appropriate one, and deduce that the ball hits the stumps after approximately half a second.

(ii) State the range of values of h for which the bowler projects the ball below the horizontal.

(iii) In the case $h = 2.5$, give an approximate value in degrees, correct to two significant figures, for α. You need not justify the accuracy of your approximation.

[You may use the small-angle approximations $\cos\theta \approx 1$ and $\sin\theta \approx \theta$.]

10 The lengths of the sides of a rectangular billiards table $ABCD$ are given by $AB = DC = a$ and $AD = BC = 2b$. There are small pockets at the midpoints M and N of the sides AD and BC, respectively. The sides of the table may be taken as smooth vertical walls.

A small ball is projected along the table from the corner A. It strikes the side BC at X, then the side DC at Y and then goes directly into the pocket at M. The angles BAX, CXY and DYM are α, β and γ respectively. On each stage of its path, the ball moves with constant speed in a straight line, the speeds being u, v and w respectively. The coefficient of restitution between the ball and the sides is e, where $e > 0$.

(i) Show that $\tan\alpha\tan\beta = e$ and find γ in terms of α.

(ii) Show that $\tan\alpha = \dfrac{(1+2e)b}{(1+e)a}$ and deduce that the shot is possible whatever the value of e.

(iii) Find an expression in terms of e for the fraction of the kinetic energy of the ball that is lost during the motion.

11 A wedge of mass km has the shape (in cross-section) of a right-angled triangle. It stands on a smooth horizontal surface with one face vertical. The inclined face makes an angle θ with the horizontal surface. A particle P, of mass m, is placed on the inclined face and released from rest. The horizontal face of the wedge is smooth, but the inclined face is rough and the coefficient of friction between P and this face is μ.

(i) When P is released, it slides down the inclined plane at an acceleration a relative to the wedge. Show that the acceleration of the wedge is

$$\frac{a\cos\theta}{k+1}\,.$$

To a stationary observer, P appears to descend along a straight line inclined at an angle $45°$ to the horizontal. Show that

$$\tan\theta = \frac{k}{k+1}\,.$$

In the case $k = 3$, find an expression for a in terms of g and μ.

(ii) What happens when P is released if $\tan\theta \leqslant \mu$?

Section C: Probability and Statistics

12 In the High Court of Farnia, the outcome of each case is determined by three judges: the ass, the beaver and the centaur. Each judge decides its verdict independently. Being simple creatures, they make their decisions entirely at random. Past verdicts show that the ass gives a guilty verdict with probability p, the beaver gives a guilty verdict with probability $p/3$ and the centaur gives a guilty verdict with probability p^2.

Let X be the number of guilty verdicts given by the three judges in a case. Given that $\mathrm{E}(X) = 4/3$, find the value of p.

The probability that a defendant brought to trial is guilty is t. The King pronounces that the defendant is guilty if at least two of the judges give a guilty verdict; otherwise, he pronounces the defendant not guilty. Find the value of t such that the probability that the King pronounces correctly is $1/2$.

13 Bag P and bag Q each contain n counters, where $n \geqslant 2$. The counters are identical in shape and size, but coloured either black or white. First, k counters ($0 \leqslant k \leqslant n$) are drawn at random from bag P and placed in bag Q. Then, k counters are drawn at random from bag Q and placed in bag P.

(i) If initially $n-1$ counters in bag P are white and one is black, and all n counters in bag Q are white, find the probability in terms of n and k that the black counter ends up in bag P.

Find the value or values of k for which this probability is maximised.

(ii) If initially $n-1$ counters in bag P are white and one is black, and $n-1$ counters in bag Q are white and one is black, find the probability in terms of n and k that the black counters end up in the same bag.

Find the value or values of k for which this probability is maximised.