Section A: Pure Mathematics

1 The points S, T, U and V have coordinates (s, ms), (t, mt), (u, nu) and (v, nv), respectively. The lines SV and UT meet the line $y = 0$ at the points with coordinates $(p, 0)$ and $(q, 0)$, respectively. Show that

$$p = \frac{(m-n)sv}{ms - nv} ,$$

and write down a similar expression for q.

Given that S and T lie on the circle $x^2 + (y-c)^2 = r^2$, find a quadratic equation satisfied by s and by t, and hence determine st and $s+t$ in terms of m, c and r.

Given that S, T, U and V lie on the above circle, show that $p + q = 0$.

2 **(i)** Let $y = \sum_{n=0}^{\infty} a_n x^n$, where the coefficients a_n are independent of x and are such that this series and all others in this question converge. Show that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} ,$$

and write down a similar expression for y''.

Write out explicitly each of the three series as far as the term containing a_3.

(ii) It is given that y satisfies the differential equation

$$xy'' - y' + 4x^3 y = 0 .$$

By substituting the series of part (i) into the differential equation and comparing coefficients, show that $a_1 = 0$.

Show that, for $n \geqslant 4$,

$$a_n = -\frac{4}{n(n-2)}\, a_{n-4} ,$$

and that, if $a_0 = 1$ and $a_2 = 0$, then $y = \cos(x^2)$.

Find the corresponding result when $a_0 = 0$ and $a_2 = 1$.

3 The function $\mathrm{f}(t)$ is defined, for $t \neq 0$, by

$$\mathrm{f}(t) = \frac{t}{\mathrm{e}^t - 1}\,.$$

(i) By expanding e^t, show that $\lim\limits_{t\to 0} \mathrm{f}(t) = 1$. Find $\mathrm{f}'(t)$ and evaluate $\lim\limits_{t\to 0} \mathrm{f}'(t)$.

(ii) Show that $\mathrm{f}(t) + \frac{1}{2}t$ is an even function. [**Note:** A function $\mathrm{g}(t)$ is said to be *even* if $\mathrm{g}(t) \equiv \mathrm{g}(-t)$.]

(iii) Show with the aid of a sketch that $\mathrm{e}^t(1-t) \leqslant 1$ and deduce that $\mathrm{f}'(t) \neq 0$ for $t \neq 0$.

Sketch the graph of $\mathrm{f}(t)$.

4 For any given (suitable) function f, the *Laplace transform* of f is the function F defined by

$$\mathrm{F}(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{f}(t)\,\mathrm{d}t \qquad (s > 0)\,.$$

(i) Show that the Laplace transform of $\mathrm{e}^{-bt}\mathrm{f}(t)$, where $b > 0$, is $\mathrm{F}(s+b)$.

(ii) Show that the Laplace transform of $\mathrm{f}(at)$, where $a > 0$, is $a^{-1}\mathrm{F}(\frac{s}{a})$.

(iii) Show that the Laplace transform of $\mathrm{f}'(t)$ is $s\mathrm{F}(s) - \mathrm{f}(0)$.

(iv) In the case $\mathrm{f}(t) = \sin t$, show that $\mathrm{F}(s) = \dfrac{1}{s^2+1}$.

Using only these four results, find the Laplace transform of $\mathrm{e}^{-pt}\cos qt$, where $p > 0$ and $q > 0$.

5 The numbers x, y and z satisfy

$$\begin{aligned} x+y+z&=1\\ x^2+y^2+z^2&=2\\ x^3+y^3+z^3&=3\,. \end{aligned}$$

Show that

$$yz+zx+xy=-\frac{1}{2}\,.$$

Show also that $x^2y+x^2z+y^2z+y^2x+z^2x+z^2y=-1$, and hence that

$$xyz=\frac{1}{6}\,.$$

Let $S_n=x^n+y^n+z^n$. Use the above results to find numbers a, b and c such that the relation

$$S_{n+1}=aS_n+bS_{n-1}+cS_{n-2}\,,$$

holds for all n.

6 Show that $\left|e^{i\beta}-e^{i\alpha}\right|=2\sin\frac{1}{2}(\beta-\alpha)$ for $0<\alpha<\beta<2\pi$. Hence show that

$$\left|e^{i\alpha}-e^{i\beta}\right|\,\left|e^{i\gamma}-e^{i\delta}\right|+\left|e^{i\beta}-e^{i\gamma}\right|\,\left|e^{i\alpha}-e^{i\delta}\right|=\left|e^{i\alpha}-e^{i\gamma}\right|\,\left|e^{i\beta}-e^{i\delta}\right|,$$

where $0<\alpha<\beta<\gamma<\delta<2\pi$.

Interpret this result as a theorem about cyclic quadrilaterals.

7 **(i)** The functions $\mathrm{f}_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$\mathrm{f}_0(x) = \frac{1}{1+x^2} \qquad \text{and} \qquad \mathrm{f}_{n+1}(x) = \frac{\mathrm{d}\mathrm{f}_n(x)}{\mathrm{d}x}\,.$$

Prove, for $n \geqslant 1$, that

$$(1+x^2)\mathrm{f}_{n+1}(x) + 2(n+1)x\mathrm{f}_n(x) + n(n+1)\mathrm{f}_{n-1}(x) = 0\,.$$

(ii) The functions $\mathrm{P}_n(x)$ are defined for $n = 0, 1, 2, \ldots$, by

$$\mathrm{P}_n(x) = (1+x^2)^{n+1}\mathrm{f}_n(x)\,.$$

Find expressions for $\mathrm{P}_0(x)$, $\mathrm{P}_1(x)$ and $\mathrm{P}_2(x)$.

Prove, for $n \geqslant 0$, that

$$\mathrm{P}_{n+1}(x) - (1+x^2)\frac{\mathrm{d}\mathrm{P}_n(x)}{\mathrm{d}x} + 2(n+1)x\mathrm{P}_n(x) = 0\,,$$

and that $\mathrm{P}_n(x)$ is a polynomial of degree n.

8 Let m be a positive integer and let n be a non-negative integer.

(i) Use the result $\lim\limits_{t\to\infty} \mathrm{e}^{-mt}t^n = 0$ to show that

$$\lim_{x\to 0} x^m(\ln x)^n = 0\,.$$

By writing x^x as $\mathrm{e}^{x\ln x}$ show that

$$\lim_{x\to 0} x^x = 1\,.$$

(ii) Let $I_n = \displaystyle\int_0^1 x^m(\ln x)^n \mathrm{d}x$. Show that

$$I_{n+1} = -\frac{n+1}{m+1}I_n$$

and hence evaluate I_n.

(iii) Show that

$$\int_0^1 x^x \mathrm{d}x = 1 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^3 - \left(\tfrac{1}{4}\right)^4 + \cdots\,.$$

Section B: Mechanics

9 A particle is projected under gravity from a point P and passes through a point Q. The angles of the trajectory with the positive horizontal direction at P and at Q are θ and ϕ, respectively. The angle of elevation of Q from P is α.

(i) Show that $\tan\theta + \tan\phi = 2\tan\alpha$.

(ii) It is given that there is a second trajectory from P to Q with the same speed of projection. The angles of this trajectory with the positive horizontal direction at P and at Q are θ' and ϕ', respectively. By considering a quadratic equation satisfied by $\tan\theta$, show that $\tan(\theta+\theta') = -\cot\alpha$. Show also that $\theta+\theta' = \pi+\phi+\phi'$.

10 A light spring is fixed at its lower end and its axis is vertical. When a certain particle P rests on the top of the spring, the compression is d. When, instead, P is dropped onto the top of the spring from a height h above it, the compression at time t after P hits the top of the spring is x. Obtain a second-order differential equation relating x and t for $0 \leqslant t \leqslant T$, where T is the time at which P first loses contact with the spring.

Find the solution of this equation in the form

$$x = A + B\cos(\omega t) + C\sin(\omega t)\,,$$

where the constants A, B, C and ω are to be given in terms of d, g and h as appropriate.

Show that

$$T = \sqrt{d/g}\,\left(2\pi - 2\arctan\sqrt{2h/d}\,\right)\,.$$

11 A comet in deep space picks up mass as it travels through a large stationary dust cloud. It is subject to a gravitational force of magnitude Mf acting in the direction of its motion. When it entered the cloud, the comet had mass M and speed V. After a time t, it has travelled a distance x through the cloud, its mass is $M(1 + bx)$, where b is a positive constant, and its speed is v.

(i) In the case when $f = 0$, write down an equation relating V, x, v and b. Hence find an expression for x in terms of b, V and t.

(ii) In the case when f is a non-zero constant, use Newton's second law in the form

force $=$ rate of change of momentum

to show that

$$v = \frac{ft + V}{1 + bx}\,.$$

Hence find an expression for x in terms of b, V, f and t.

Show that it is possible, if b, V and f are suitably chosen, for the comet to move with constant speed. Show also that, if the comet does not move with constant speed, its speed tends to a constant as $t \to \infty$.

Section C: Probability and Statistics

12 **(i)** Albert tosses a fair coin k times, where k is a given positive integer. The number of heads he gets is X_1. He then tosses the coin X_1 times, getting X_2 heads. He then tosses the coin X_2 times, getting X_3 heads. The random variables X_4, X_5, $\dots$ are defined similarly. Write down $\mathrm{E}(X_1)$.

By considering $\mathrm{E}(X_2 \mid X_1 = x_1)$, or otherwise, show that $\mathrm{E}(X_2) = \frac{1}{4}k$.

Find $\displaystyle\sum_{i=1}^{\infty} \mathrm{E}(X_i)$.

(ii) Bertha has k fair coins. She tosses the first coin until she gets a tail. The number of heads she gets before the first tail is Y_1. She then tosses the second coin until she gets a tail and the number of heads she gets with this coin before the first tail is Y_2. The random variables $Y_3, Y_4, \dots, Y_k$ are defined similarly, and $Y = \sum_{i=1}^{k} Y_i$.

Obtain the probability generating function of Y, and use it to find $\mathrm{E}(Y)$, $\mathrm{Var}\,(Y)$ and $\mathrm{P}(Y = r)$.

13 **(i)** The point P lies on the circumference of a circle of unit radius and centre O. The angle, θ, between OP and the positive x-axis is a random variable, uniformly distributed on the interval $0 \leqslant \theta < 2\pi$. The cartesian coordinates of P with respect to O are (X, Y). Find the probability density function for X, and calculate $\mathrm{Var}\,(X)$.

Show that X and Y are uncorrelated and discuss briefly whether they are independent.

(ii) The points P_i ($i = 1, 2, \dots, n$) are chosen independently on the circumference of the circle, as in part (i), and have cartesian coordinates (X_i, Y_i). The point $\overline{P}$ has coordinates $(\overline{X}, \overline{Y})$, where $\overline{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$ and $\overline{Y} = \dfrac{1}{n}\sum_{i=1}^{n} Y_i$. Show that $\overline{X}$ and $\overline{Y}$ are uncorrelated.

Show that, for large n, $\mathrm{P}\left(|\overline{X}| \leqslant \sqrt{\dfrac{2}{n}}\right) \approx 0.95$.