Section A: Pure Mathematics

1 Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a\,(x - y + 2)^2 + b\,(cx + y)^2 + d\,,$$

find the values of the constants a, b, c and d.

Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9\,,$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$

2 The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, where a and b are constants, has two stationary points. Show that

$$a - b < 0 \quad \text{or} \quad a - b > 4\,.$$

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

3 Show that

$$\sin(x + y) - \sin(x - y) = 2\cos x\,\sin y$$

and deduce that

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A + B)\,\sin\tfrac{1}{2}(A - B)\,.$$

Show also that

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A + B)\,\sin\tfrac{1}{2}(A - B)\,.$$

The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ and $(a\cos s, b\sin s)$ respectively, where $0 \leqslant p < q < r < s < 2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r + s - p - q = 2\pi\,.$$

4 Use the substitution $x = \dfrac{1}{t^2-1}$, where $t > 1$, to show that, for $x > 0$,

$$\int \frac{1}{\sqrt{x\,(x+1)}}\ \mathrm{d}x = 2\ln\left(\sqrt{x} + \sqrt{x+1}\ \right) + c\,.$$

[Note You may use without proof the result $\displaystyle\int \frac{1}{t^2-a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \text{constant.}$ **]**

The section of the curve

$$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through 360^o about the x-axis. Show that the volume enclosed is $2\pi\ln\frac{5}{4}$.

5 By considering the expansion of $(1+x)^n$ where n is a positive integer, or otherwise, show that:

(i) $\dbinom{n}{0} + \dbinom{n}{1} + \dbinom{n}{2} + \cdots + \dbinom{n}{n} = 2^n\,;$

(ii) $\dbinom{n}{1} + 2\dbinom{n}{2} + 3\dbinom{n}{3} + \cdots + n\dbinom{n}{n} = n2^{n-1}\,;$

(iii) $\dbinom{n}{0} + \dfrac{1}{2}\dbinom{n}{1} + \dfrac{1}{3}\dbinom{n}{2} + \cdots + \dfrac{1}{n+1}\dbinom{n}{n} = \dfrac{1}{n+1}\left(2^{n+1}-1\right)\,;$

(iv) $\dbinom{n}{1} + 2^2\dbinom{n}{2} + 3^2\dbinom{n}{3} + \cdots + n^2\dbinom{n}{n} = n\,(n+1)\,2^{n-2}\,.$

6 Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

7 Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}\,,$$

where α and β are positive constants with $\alpha+\beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$ and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta}\,,$$

and write down $\mathbf{q}$ in terms of $\alpha,\ \beta$ and $\mathbf{b}$.

Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a}+\beta\mathbf{b}}{\alpha+\beta}\,,$$

lies on the lines OC and AB.

The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

8 **(i)** Suppose that a, b and c are integers that satisfy the equation

$$a^3 + 3b^3 = 9c^3.$$

Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.

(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4 + 2q^4 = 5r^4\,.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.

Section B: Mechanics

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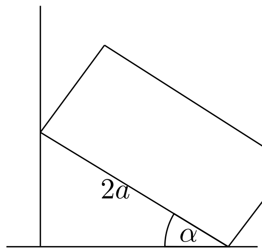

The diagram shows a uniform rectangular lamina with sides of lengths $2a$ and $2b$ leaning against a rough vertical wall, with one corner resting on a rough horizontal plane. The plane of the lamina is vertical and perpendicular to the wall, and one edge makes an angle of α with the horizontal plane. Show that the centre of mass of the lamina is a distance $a\cos\alpha + b\sin\alpha$ from the wall.

The coefficients of friction at the two points of contact are each μ and the friction is limiting at both contacts. Show that

$$a\cos(2\lambda + \alpha) = b\sin\alpha\,,$$

where $\tan\lambda = \mu$.

Show also that if the lamina is square, then $\lambda = \frac{1}{4}\pi - \alpha$.

10 A particle P moves so that, at time t, its displacement $\mathbf{r}$ from a fixed origin is given by

$$\mathbf{r} = \left(\mathrm{e}^t\cos t\right)\mathbf{i} + \left(\mathrm{e}^t\sin t\right)\mathbf{j}\,.$$

Show that the velocity of the particle always makes an angle of $\frac{\pi}{4}$ with the particle's displacement, and that the acceleration of the particle is always perpendicular to its displacement. Sketch the path of the particle for $0 \leqslant t \leqslant \pi$.

A second particle Q moves on the same path, passing through each point on the path a fixed time T after P does. Show that the distance between P and Q is proportional to e^t.

11 Two particles of masses m and M, with $M > m$, lie in a smooth circular groove on a horizontal plane. The coefficient of restitution between the particles is e. The particles are initially projected round the groove with the same speed u but in opposite directions. Find the speeds of the particles after they collide for the first time and show that they will both change direction if $2em > M - m$.

After a further $2n$ collisions, the speed of the particle of mass m is v and the speed of the particle of mass M is V. Given that at each collision both particles change their directions of motion, explain why

$$mv - MV = u(M - m),$$

and find v and V in terms of m, M, e, u and n.

Section C: Probability and Statistics

12 A discrete random variable X takes only positive integer values. Define $\mathrm{E}(X)$ for this case, and show that

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} \mathrm{P}\left(X \geqslant n\right).$$

I am collecting toy penguins from cereal boxes. Each box contains either one daddy penguin or one mummy penguin. The probability that a given box contains a daddy penguin is p and the probability that a given box contains a mummy penguin is q, where $p \neq 0$, $q \neq 0$ and $p + q = 1$.

Let X be the number of boxes that I need to open to get at least one of each kind of penguin. Show that $\mathrm{P}(X \geqslant 4) = p^3 + q^3$, and that

$$\mathrm{E}(X) = \frac{1}{pq} - 1.$$

Hence show that $\mathrm{E}(X) \geqslant 3$.

13 The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$p\mathrm{e}^{2\lambda} - \mathrm{e}^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.