# Section A: Pure Mathematics

**1** Let $x_1$, $x_2$, ..., $x_n$ and $x_{n+1}$ be any fixed real numbers. The numbers $A$ and $B$ are defined by

$$A = \frac{1}{n}\sum_{k=1}^{n} x_k\,, \quad B = \frac{1}{n}\sum_{k=1}^{n}(x_k - A)^2\,,$$

and the numbers $C$ and $D$ are defined by

$$C = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k\,, \quad D = \frac{1}{n+1}\sum_{k=1}^{n+1}(x_k - C)^2\,.$$

**(i)** Express $C$ in terms of $A$, $x_{n+1}$ and $n$.

**(ii)** Show that $B = \dfrac{1}{n}\displaystyle\sum_{k=1}^{n} x_k^2 - A^2$.

**(iii)** Express $D$ in terms of $B$, $A$, $x_{n+1}$ and $n$.

Hence show that $(n+1)D \geqslant nB$ for all values of $x_{n+1}$, but that $D < B$ if and only if

$$A - \sqrt{\frac{(n+1)B}{n}} < x_{n+1} < A + \sqrt{\frac{(n+1)B}{n}}\,.$$

**2** In this question, $a$ is a positive constant.

**(i)** Express $\cosh a$ in terms of exponentials.

By using partial fractions, prove that

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,\mathrm{d}x = \frac{a}{2\sinh a}\,.$$

**(ii)** Find, expressing your answers in terms of hyperbolic functions,

$$\int_1^\infty \frac{1}{x^2 + 2x\sinh a - 1}\,\mathrm{d}x$$

and

$$\int_0^\infty \frac{1}{x^4 + 2x^2\cosh a + 1}\,\mathrm{d}x\,.$$

**3** For any given positive integer $n$, a number $a$ (which may be complex) is said to be a *primitive $n$th root of unity* if $a^n = 1$ and there is no integer $m$ such that $0 < m < n$ and $a^m = 1$. Write down the two primitive 4th roots of unity.

Let $\mathrm{C}_n(x)$ be the polynomial such that the roots of the equation $\mathrm{C}_n(x) = 0$ are the primitive $n$th roots of unity, the coefficient of the highest power of $x$ is one and the equation has no repeated roots. Show that $\mathrm{C}_4(x) = x^2 + 1$.

**(i)** Find $\mathrm{C}_1(x)$, $\mathrm{C}_2(x)$, $\mathrm{C}_3(x)$, $\mathrm{C}_5(x)$ and $\mathrm{C}_6(x)$, giving your answers as unfactorised polynomials.

**(ii)** Find the value of $n$ for which $\mathrm{C}_n(x) = x^4 + 1$.

**(iii)** Given that $p$ is prime, find an expression for $\mathrm{C}_p(x)$, giving your answer as an unfactorised polynomial.

**(iv)** Prove that there are no positive integers $q$, $r$ and $s$ such that $\mathrm{C}_q(x) \equiv \mathrm{C}_r(x)\mathrm{C}_s(x)$.

**4** **(i)** The number $\alpha$ is a common root of the equations $x^2 + ax + b = 0$ and $x^2 + cx + d = 0$ (that is, $\alpha$ satisfies both equations). Given that $a \neq c$, show that

$$\alpha = -\frac{b-d}{a-c}\,.$$

Hence, or otherwise, show that the equations have at least one common root if and only if

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0\,.$$

Does this result still hold if the condition $a \neq c$ is not imposed?

**(ii)** Show that the equations $x^2 + ax + b = 0$ and $x^3 + (a+1)x^2 + qx + r = 0$ have at least one common root if and only if

$$(b-r)^2 - a(b-r)(a+b-q) + b(a+b-q)^2 = 0\,.$$

Hence, or otherwise, find the values of $b$ for which the equations $2x^2 + 5x + 2b = 0$ and $2x^3 + 7x^2 + 5x + 1 = 0$ have at least one common root.

**5** The vertices $A$, $B$, $C$ and $D$ of a square have coordinates $(0,0)$, $(a,0)$, $(a,a)$ and $(0,a)$, respectively. The points $P$ and $Q$ have coordinates $(an,0)$ and $(0,am)$ respectively, where $0 < m < n < 1$. The line $CP$ produced meets $DA$ produced at $R$ and the line $CQ$ produced meets $BA$ produced at $S$. The line $PQ$ produced meets the line $RS$ produced at $T$. Show that $TA$ is perpendicular to $AC$.

Explain how, given a square of area $a^2$, a square of area $2a^2$ may be constructed using only a straight-edge.

[**Note**: a straight-edge is a ruler with no markings on it; no measurements (and no use of compasses) are allowed in the construction.]

**6** The points $P$, $Q$ and $R$ lie on a sphere of unit radius centred at the origin, $O$, which is fixed. Initially, $P$ is at $P_0(1,0,0)$, $Q$ is at $Q_0(0,1,0)$ and $R$ is at $R_0(0,0,1)$.

**(i)** The sphere is then rotated about the $z$-axis, so that the line $OP$ turns directly towards the positive $y$-axis through an angle $\phi$. The position of $P$ after this rotation is denoted by $P_1$. Write down the coordinates of $P_1$.

**(ii)** The sphere is now rotated about the line in the $x$-$y$ plane perpendicular to $OP_1$, so that the line $OP$ turns directly towards the positive $z$-axis through an angle $\lambda$. The position of $P$ after this rotation is denoted by $P_2$. Find the coordinates of $P_2$. Find also the coordinates of the points $Q_2$ and $R_2$, which are the positions of $Q$ and $R$ after the two rotations.

**(iii)** The sphere is now rotated for a third time, so that $P$ returns from $P_2$ to its original position $P_0$. During the rotation, $P$ remains in the plane containing $P_0$, $P_2$ and $O$. Show that the angle of this rotation, $\theta$, satisfies

$$\cos\theta = \cos\phi\cos\lambda\,,$$

and find a vector in the direction of the axis about which this rotation takes place.

**7** Given that $y = \cos(m \arcsin x)$, for $|x| < 1$, prove that

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + m^2y = 0\,.$$

Obtain a similar equation relating $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, and a similar equation relating $\dfrac{\mathrm{d}^4y}{\mathrm{d}x^4}$, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

Conjecture and prove a relation between $\dfrac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}$, $\dfrac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}$ and $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$.

Obtain the first three non-zero terms of the Maclaurin series for $y$. Show that, if $m$ is an even integer, $\cos m\theta$ may be written as a polynomial in $\sin\theta$ beginning

$$1 - \frac{m^2\sin^2\theta}{2!} + \frac{m^2(m^2-2^2)\sin^4\theta}{4!} - \cdots\,. \qquad (|\theta| < \tfrac{1}{2}\pi)$$

State the degree of the polynomial.

**8** Given that $\mathrm{P}(x) = \mathrm{Q}(x)\mathrm{R}'(x) - \mathrm{Q}'(x)\mathrm{R}(x)$, write down an expression for

$$\int \frac{\mathrm{P}(x)}{\big(\mathrm{Q}(x)\big)^2}\,\mathrm{d}x\,.$$

**(i)** By choosing the function $\mathrm{R}(x)$ to be of the form $a + bx + cx^2$, find

$$\int \frac{5x^2-4x-3}{(1+2x+3x^2)^2}\,\mathrm{d}x\,.$$

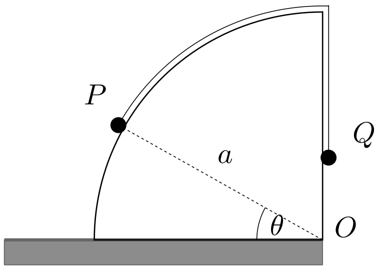
Show that the choice of $\mathrm{R}(x)$ is not unique and, by comparing the two functions $\mathrm{R}(x)$ corresponding to two different values of $a$, explain how the different choices are related.

**(ii)** Find the general solution of

$$(1+\cos x + 2\sin x)\frac{\mathrm{d}y}{\mathrm{d}x} + (\sin x - 2\cos x)y = 5 - 3\cos x + 4\sin x\,.$$

## Section B: Mechanics

**9**




The diagram shows two particles, $P$ and $Q$, connected by a light inextensible string which passes over a smooth block fixed to a horizontal table. The cross-section of the block is a quarter circle with centre $O$, which is at the edge of the table, and radius $a$. The angle between $OP$ and the table is $\theta$. The masses of $P$ and $Q$ are $m$ and $M$, respectively, where $m < M$.

Initially, $P$ is held at rest on the table and in contact with the block, $Q$ is vertically above $O$, and the string is taut. Then $P$ is released. Given that, in the subsequent motion, $P$ remains in contact with the block as $\theta$ increases from $0$ to $\frac{1}{2}\pi$, find an expression, in terms of $m$, $M$, $\theta$ and $g$, for the normal reaction of the block on $P$ and show that

$$\frac{m}{M} \geqslant \frac{\pi - 1}{3}\,.$$

**10** A small bead $B$, of mass $m$, slides without friction on a fixed horizontal ring of radius $a$. The centre of the ring is at $O$. The bead is attached by a light elastic string to a fixed point $P$ in the plane of the ring such that $OP = b$, where $b > a$. The natural length of the elastic string is $c$, where $c < b - a$, and its modulus of elasticity is $\lambda$. Show that the equation of motion of the bead is

$$ma\ddot{\phi} = -\lambda\left(\frac{a\sin\phi}{c\sin\theta} - 1\right)\sin(\theta + \phi)\,,$$

where $\theta = \angle BPO$ and $\phi = \angle BOP$.

Given that $\theta$ and $\phi$ are small, show that $a(\theta + \phi) \approx b\theta$. Hence find the period of small oscillations about the equilibrium position $\theta = \phi = 0$.

**11** A bullet of mass $m$ is fired horizontally with speed $u$ into a wooden block of mass $M$ at rest on a horizontal surface. The coefficient of friction between the block and the surface is $\mu$. While the bullet is moving through the block, it experiences a constant force of resistance to its motion of magnitude $R$, where $R > (M+m)\mu g$. The bullet moves horizontally in the block and does not emerge from the other side of the block.

**(i)** Show that the magnitude, $a$, of the deceleration of the bullet relative to the block while the bullet is moving through the block is given by

$$a = \frac{R}{m} + \frac{R - (M+m)\mu g}{M}\,.$$

**(ii)** Show that the common speed, $v$, of the block and bullet when the bullet stops moving through the block satisfies

$$av = \frac{Ru - (M+m)\mu g u}{M}\,.$$

**(iii)** Obtain an expression, in terms of $u$, $v$ and $a$, for the distance moved by the block while the bullet is moving through the block.

**(iv)** Show that the total distance moved by the block is

$$\frac{muv}{2(M+m)\mu g}\,.$$

Describe briefly what happens if $R < (M+m)\mu g$.

## Section C: Probability and Statistics

**12** The infinite series $S$ is given by

$$S = 1 + (1+d)r + (1+2d)r^2 + \cdots + (1+nd)r^n + \cdots \ ,$$

for $|r| < 1$. By considering $S - rS$, or otherwise, prove that

$$S = \frac{1}{1-r} + \frac{rd}{(1-r)^2} \, .$$

Arthur and Boadicea shoot arrows at a target. The probability that an arrow shot by Arthur hits the target is $a$; the probability that an arrow shot by Boadicea hits the target is $b$. Each shot is independent of all others. Prove that the expected number of shots it takes Arthur to hit the target is $1/a$.

Arthur and Boadicea now have a contest. They take alternate shots, with Arthur going first. The winner is the one who hits the target first. The probability that Arthur wins the contest is $\alpha$ and the probability that Boadicea wins is $\beta$. Show that

$$\alpha = \frac{a}{1 - a'b'} \, ,$$

where $a' = 1 - a$ and $b' = 1 - b$, and find $\beta$.

Show that the expected number of shots in the contest is $\dfrac{\alpha}{a} + \dfrac{\beta}{b}$.

**13** In this question, $\mathrm{Corr}(U, V)$ denotes the product moment correlation coefficient between the random variables $U$ and $V$, defined by

$$\mathrm{Corr}(U, V) \equiv \frac{\mathrm{Cov}(U, V)}{\sqrt{\mathrm{Var}\,(U)\mathrm{Var}\,(V)}} \, .$$

The independent random variables $Z_1$, $Z_2$ and $Z_3$ each have expectation 0 and variance 1. What is the value of $\mathrm{Corr}(Z_1, Z_2)$?

Let $Y_1 = Z_1$ and let

$$Y_2 = \rho_{12} Z_1 + (1 - \rho_{12}^2)^{\frac{1}{2}} Z_2 \, ,$$

where $\rho_{12}$ is a given constant with $-1 < \rho_{12} < 1$. Find $\mathrm{E}(Y_2)$, $\mathrm{Var}\,(Y_2)$ and $\mathrm{Corr}(Y_1, Y_2)$.

Now let $Y_3 = aZ_1 + bZ_2 + cZ_3$, where $a$, $b$ and $c$ are real constants and $c \geqslant 0$. Given that $\mathrm{E}(Y_3) = 0$, $\mathrm{Var}\,(Y_3) = 1$, $\mathrm{Corr}(Y_1, Y_3) = \rho_{13}$ and $\mathrm{Corr}(Y_2, Y_3) = \rho_{23}$, express $a$, $b$ and $c$ in terms of $\rho_{23}$, $\rho_{13}$ and $\rho_{12}$.

Given constants $\mu_i$ and $\sigma_i$, for $i = 1$, $2$ and $3$, give expressions in terms of the $Y_i$ for random variables $X_i$ such that $\mathrm{E}(X_i) = \mu_i$, $\mathrm{Var}\,(X_i) = \sigma_i^2$ and $\mathrm{Corr}(X_i, X_j) = \rho_{ij}$.