Section A: Pure Mathematics

1 **(i)** Show that the gradient of the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $b \neq 0$, is $-\dfrac{ay^2}{bx^2}$.

The point (p, q) lies on both the straight line $ax + by = 1$ and the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $ab \neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.

Show further that either $(a - b)^2 = 1$ or $(a + b)^2 = 1$.

(ii) Show that if the straight line $ax + by = 1$, where $ab \neq 0$, is a normal to the curve $\dfrac{a}{x} - \dfrac{b}{y} = 1$, then $a^2 - b^2 = \frac{1}{2}$.

2 The number E is defined by $E = \displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x$.

Show that

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x = \mathrm{e} - 1 - E\,,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ in terms of e and E.

Evaluate also, in terms of E and e as appropriate:

(i) $\displaystyle\int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x\,.$

3 Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,. \qquad (*)$$

(i) By differentiating $(*)$, or otherwise, show that

$$\cot\tfrac{1}{9}\pi-\cot\tfrac{2}{9}\pi+\cot\tfrac{4}{9}\pi=\sqrt{3}\,.$$

(ii) By setting $\theta=\frac{1}{6}\pi-\phi$ in $(*)$, or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that

$$\cot\theta\cot(\tfrac{1}{3}\pi-\theta)\cot(\tfrac{1}{3}\pi+\theta)=\cot 3\theta\,.$$

Show that

$$\operatorname{cosec}\tfrac{1}{9}\pi-\operatorname{cosec}\tfrac{5}{9}\pi+\operatorname{cosec}\tfrac{7}{9}\pi=2\sqrt{3}\,.$$

4 The distinct points P and Q, with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2=4ax$. The tangents to the curve at P and Q meet at the point T. Show that T has coordinates $\big(apq, a(p+q)\big)$. You may assume that $p\neq 0$ and $q\neq 0$.

The point F has coordinates $(a, 0)$ and ϕ is the angle TFP. Show that

$$\cos\phi=\frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$

and deduce that the line FT bisects the angle PFQ.

5 Given that $0<k<1$, show with the help of a sketch that the equation

$$\sin x=kx \qquad (*)$$

has a unique solution in the range $0<x<\pi$.

Let

$$I=\int_0^{\pi}\big|\sin x-kx\big|\,\mathrm{d}x\,.$$

Show that

$$I=\frac{\pi^2\sin\alpha}{2\alpha}-2\cos\alpha-\alpha\sin\alpha\,,$$

where α is the unique solution of $(*)$.

Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of I is

$$-2\cos\frac{\pi}{\sqrt{2}}\,.$$

6 Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of

$$\frac{1-x+2x^2}{(1-x)^3}$$

is r^2+1 and hence find the sum of the series

$$1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\frac{37}{64}+\frac{50}{128}+\cdots.$$

(ii) Find the sum of the series

$$1+2+\frac{9}{4}+2+\frac{25}{16}+\frac{9}{8}+\frac{49}{64}+\cdots.$$

7 In this question, you may assume that $\ln(1+x)\approx x-\frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t}=k(\alpha^2 H-h)\,,$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT=\ln\left(1+\frac{1}{\alpha}\right)\,,$$

and that $kT\approx\alpha^{-1}$ for large values of α.

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT'=2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(1+\frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT'\approx\sqrt{H}$ for large values of α.

8 **(i)** The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$

(a) Show that $m > n$. Show also that $m < n + 1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n + 1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m, n) = (1, 0)$ and $(m, n) = (1, -1)$.

(ii) Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1\,.$$

Section B: Mechanics

9 A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}\,.$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

10 A particle, A, is dropped from a point P which is at a height h above a horizontal plane. A second particle, B, is dropped from P and first collides with A after A has bounced on the plane and before A reaches P again. The bounce and the collision are both perfectly elastic. Explain why the speeds of A and B immediately before the first collision are the same.

The masses of A and B are M and m, respectively, where $M > 3m$, and the speed of the particles immediately before the first collision is u. Show that both particles move upwards after their first collision and that the maximum height of B above the plane after the first collision and before the second collision is

$$h + \frac{4M(M-m)u^2}{(M+m)^2 g}\,.$$

11 A thin non-uniform bar AB of length $7d$ has centre of mass at a point G, where $AG = 3d$. A light inextensible string has one end attached to A and the other end attached to B. The string is hung over a smooth peg P and the bar hangs freely in equilibrium with B lower than A. Show that

$$3\sin\alpha = 4\sin\beta\,,$$

where α and β are the angles PAB and PBA, respectively.

Given that $\cos\beta = \frac{4}{5}$ and that α is acute, find in terms of d the length of the string and show that the angle of inclination of the bar to the horizontal is $\arctan\frac{1}{7}$.

Section C: Probability and Statistics

12 I am selling raffle tickets for $£1$ per ticket. In the queue for tickets, there are m people each with a single $£1$ coin and n people each with a single $£2$ coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

(i) In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

(ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

(iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

13 In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f given by

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

(ii) Let d be the value of X such that $\mathrm{P}(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.

(iii) Given that $d = \sqrt{2}$, find a.