Section A: Pure Mathematics

1 **(i)** Sketch the curve $y = \sqrt{1-x} + \sqrt{3+x}$.

Use your sketch to show that only one real value of x satisfies

$$\sqrt{1-x} + \sqrt{3+x} = x+1\,,$$

and give this value.

(ii) Determine graphically the number of real values of x that satisfy

$$2\sqrt{1-x} = \sqrt{3+x} + \sqrt{3-x}\ .$$

Solve this equation.

2 Write down the cubes of the integers 1, 2, . . . , 10 .

The positive integers x, y and z, where $x < y$, satisfy

$$x^3 + y^3 = kz^3\,, \qquad (*)$$

where k is a given positive integer.

(i) In the case $x + y = k$, show that

$$z^3 = k^2 - 3kx + 3x^2\ .$$

Deduce that $(4z^3 - k^2)/3$ is a perfect square and that $\frac{1}{4}k^2 \leqslant z^3 < k^2$.

Use these results to find a solution of $(*)$ when $k = 20$.

(ii) By considering the case $x + y = z^2$, find two solutions of $(*)$ when $k = 19$.

3 In this question, you may assume without proof that any function f for which $\mathrm{f}'(x) \geqslant 0$ is *increasing*; that is, $\mathrm{f}(x_2) \geqslant \mathrm{f}(x_1)$ if $x_2 \geqslant x_1$.

(i) **(a)** Let $\mathrm{f}(x) = \sin x - x\cos x$. Show that $\mathrm{f}(x)$ is increasing for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and deduce that $\mathrm{f}(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

(b) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arcsin x) \geqslant 1$ for $0 \leqslant x < 1$, show that

$$\arcsin x \geqslant x \qquad\qquad (0 \leqslant x < 1).$$

(c) Let $\mathrm{g}(x) = x \operatorname{cosec} x$ for $0 < x < \frac{1}{2}\pi$. Show that g is increasing and deduce that

$$(\arcsin x)\, x^{-1} \geqslant x \operatorname{cosec} x \qquad\qquad (0 < x < 1).$$

(ii) Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\arctan x) \leqslant 1$ for $x \geqslant 0$, show by considering the function $x^{-1}\tan x$ that

$$(\tan x)(\arctan x) \geqslant x^2 \qquad\qquad (0 < x < \tfrac{1}{2}\pi).$$

4 **(i)** Find all the values of θ, in the range $0^\circ < \theta < 180^\circ$, for which $\cos\theta = \sin 4\theta$. Hence show that

$$\sin 18^\circ = \frac{1}{4}\left(\sqrt{5} - 1\right).$$

(ii) Given that

$$4\sin^2 x + 1 = 4\sin^2 2x\,,$$

find all possible values of $\sin x$, giving your answers in the form $p + q\sqrt{5}$ where p and q are rational numbers.

(iii) Hence find two values of α with $0^\circ < \alpha < 90^\circ$ for which

$$\sin^2 3\alpha + \sin^2 5\alpha = \sin^2 6\alpha\,.$$

5 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to an origin O, and O, A and B are non-collinear. The point C, with position vector $\mathbf{c}$, is the reflection of B in the line through O and A. Show that $\mathbf{c}$ can be written in the form

$$\mathbf{c} = \lambda\mathbf{a} - \mathbf{b}$$

where $\lambda = \dfrac{2\,\mathbf{a.b}}{\mathbf{a.a}}$.

The point D, with position vector $\mathbf{d}$, is the reflection of C in the line through O and B. Show that $\mathbf{d}$ can be written in the form

$$\mathbf{d} = \mu\mathbf{b} - \lambda\mathbf{a}$$

for some scalar μ to be determined.

Given that A, B and D are collinear, find the relationship between λ and μ. In the case $\lambda = -\frac{1}{2}$, determine the cosine of $\angle AOB$ and describe the relative positions of A, B and D.

6 For any given function f, let

$$I = \int [\mathrm{f}'(x)]^2\,[\mathrm{f}(x)]^n \mathrm{d}x\,, \qquad (*)$$

where n is a positive integer. Show that, if $\mathrm{f}(x)$ satisfies $\mathrm{f}''(x) = k\mathrm{f}(x)\mathrm{f}'(x)$ for some constant k, then $(*)$ can be integrated to obtain an expression for I in terms of $\mathrm{f}(x)$, $\mathrm{f}'(x)$, k and n.

(i) Verify your result in the case $\mathrm{f}(x) = \tan x$. Hence find

$$\int \frac{\sin^4 x}{\cos^8 x}\,\mathrm{d}x\ .$$

(ii) Find

$$\int \sec^2 x\,(\sec x + \tan x)^6\,\mathrm{d}x\ .$$

7 The two sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ have general terms

$$a_n = \lambda^n + \mu^n \quad \text{and} \quad b_n = \lambda^n - \mu^n\,,$$

respectively, where $\lambda = 1 + \sqrt{2}$ and $\mu = 1 - \sqrt{2}$.

(i) Show that $\displaystyle\sum_{r=0}^{n} b_r = -\sqrt{2} + \frac{1}{\sqrt{2}}\,a_{n+1}$, and give a corresponding result for $\displaystyle\sum_{r=0}^{n} a_r$.

(ii) Show that, if n is odd,

$$\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \tfrac{1}{2}b_{n+1}^2\,,$$

and give a corresponding result when n is even.

(iii) Show that, if n is even,

$$\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = 2\,,$$

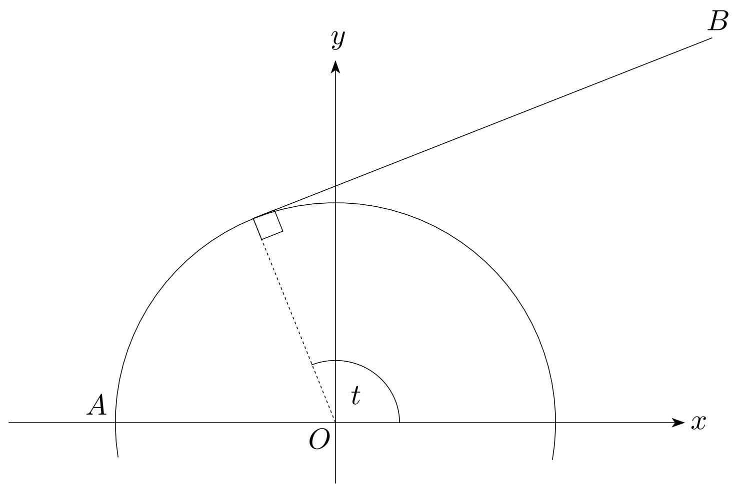
and give a corresponding result when n is odd.

8 The end A of an inextensible string AB of length π is attached to a point on the circumference of a fixed circle of unit radius and centre O. Initially the string is straight and tangent to the circle. The string is then wrapped round the circle until the end B comes into contact with the circle. The string remains taut during the motion, so that a section of the string is in contact with the circumference and the remaining section is straight.

Taking O to be the origin of cartesian coordinates with A at $(-1, 0)$ and B initially at $(-1, \pi)$, show that the curve described by B is given parametrically by

$$x = \cos t + t \sin t\,, \qquad y = \sin t - t \cos t\,,$$

where t is the angle shown in the diagram.



Find the value, t_0, of t for which x takes its maximum value on the curve, and sketch the curve.

Use the area integral $\displaystyle\int y \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ to find the area between the curve and the x axis for $\pi \geqslant t \geqslant t_0$.

Find the area swept out by the string (that is, the area between the curve described by B and the semicircle shown in the diagram).

Section B: Mechanics

9 Two particles, A of mass $2m$ and B of mass m, are moving towards each other in a straight line on a smooth horizontal plane, with speeds $2u$ and u respectively. They collide directly. Given that the coefficient of restitution between the particles is e, where $0 < e \leqslant 1$, determine the speeds of the particles after the collision.

After the collision, B collides directly with a smooth vertical wall, rebounding and then colliding directly with A for a second time. The coefficient of restitution between B and the wall is f, where $0 < f \leqslant 1$. Show that the velocity of B after its second collision with A is

$$\tfrac{2}{3}(1-e^2)u - \tfrac{1}{3}(1-4e^2)fu$$

towards the wall and that B moves towards (not away from) the wall for all values of e and f.

10 A particle is projected from a point on a horizontal plane, at speed u and at an angle θ above the horizontal. Let H be the maximum height of the particle above the plane. Derive an expression for H in terms of u, g and θ.

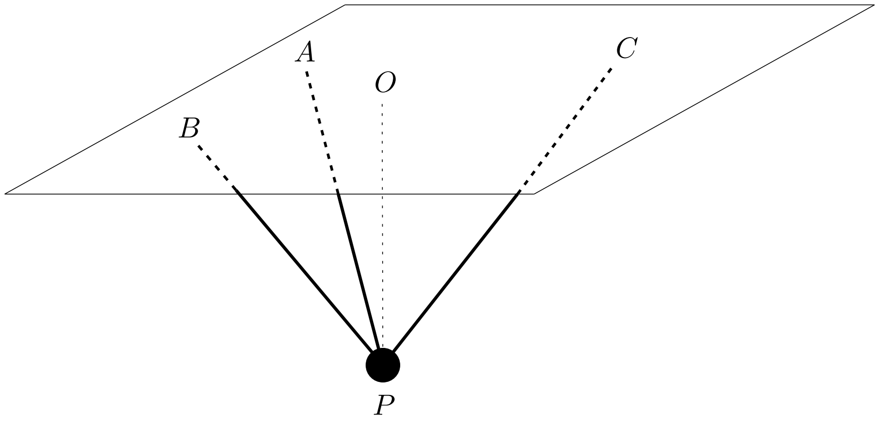
A particle P is projected from a point O on a smooth horizontal plane, at speed u and at an angle θ above the horizontal. At the same instant, a second particle R is projected horizontally from O in such a way that R is vertically below P in the ensuing motion. A light inextensible string of length $\frac{1}{2}H$ connects P and R. Show that the time that elapses before the string becomes taut is

$$(\sqrt{2}-1)\sqrt{H/g}\ .$$

When the string becomes taut, R leaves the plane, the string remaining taut. Given that P and R have equal masses, determine the total horizontal distance, D, travelled by R from the moment its motion begins to the moment it lands on the plane again, giving your answer in terms of u, g and θ.

Given that $D = H$, find the value of $\tan\theta$.

11 Three non-collinear points A, B and C lie in a horizontal ceiling. A particle P of weight W is suspended from this ceiling by means of three light inextensible strings AP, BP and CP, as shown in the diagram. The point O lies vertically above P in the ceiling.



The angles AOB and AOC are $90° + \theta$ and $90° + \phi$, respectively, where θ and ϕ are acute angles such that $\tan\theta = \sqrt{2}$ and $\tan\phi = \frac{1}{4}\sqrt{2}$.

The strings AP, BP and CP make angles $30°$, $90° - \theta$ and $60°$, respectively, with the vertical, and the tensions in these strings have magnitudes T, U and V respectively.

(i) Show that the unit vector in the direction PB can be written in the form

$$-\frac{1}{3}\,\mathbf{i} - \frac{\sqrt{2}}{3}\,\mathbf{j} + \frac{\sqrt{2}}{\sqrt{3}}\,\mathbf{k}\,,$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the usual mutually perpendicular unit vectors with $\mathbf{j}$ parallel to OA and $\mathbf{k}$ vertically upwards.

(ii) Find expressions in vector form for the forces acting on P.

(iii) Show that $U = \sqrt{6}V$ and find T, U and V in terms of W.

Section C: Probability and Statistics

12 Xavier and Younis are playing a match. The match consists of a series of games and each game consists of three points.

Xavier has probability p and Younis has probability $1 - p$ of winning the first point of any game. In the second and third points of each game, the player who won the previous point has probability p and the player who lost the previous point has probability $1 - p$ of winning the point. If a player wins two consecutive points in a single game, the match ends and that player has won; otherwise the match continues with another game.

(i) Let w be the probability that Younis wins the match. Show that, for $p \neq 0$,

$$w = \frac{1 - p^2}{2 - p}.$$

Show that $w > \frac{1}{2}$ if $p < \frac{1}{2}$, and $w < \frac{1}{2}$ if $p > \frac{1}{2}$. Does w increase whenever p decreases?

(ii) If Xavier wins the match, Younis gives him $£1$; if Younis wins the match, Xavier gives him $£k$. Find the value of k for which the game is 'fair' in the case when $p = \frac{2}{3}$.

(iii) What happens when $p = 0$?

13 What property of a distribution is measured by its *skewness*?

(i) One measure of skewness, γ, is given by

$$\gamma = \frac{\mathrm{E}\big((X-\mu)^3\big)}{\sigma^3}\,,$$

where μ and σ^2 are the mean and variance of the random variable X. Show that

$$\gamma = \frac{\mathrm{E}(X^3) - 3\mu\sigma^2 - \mu^3}{\sigma^3}\,.$$

The continuous random variable X has probability density function f where

$$\mathrm{f}(x) = \begin{cases} 2x & \text{for } 0 \leqslant x \leqslant 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Show that for this distribution $\gamma = -\dfrac{2\sqrt{2}}{5}$.

(ii) The *decile skewness*, D, of a distribution is defined by

$$D = \frac{\mathrm{F}^{-1}(\frac{9}{10}) - 2\mathrm{F}^{-1}(\frac{1}{2}) + \mathrm{F}^{-1}(\frac{1}{10})}{\mathrm{F}^{-1}(\frac{9}{10}) - \mathrm{F}^{-1}(\frac{1}{10})}\,,$$

where F^{-1} is the inverse of the cumulative distribution function. Show that, for the above distribution, $D = 2 - \sqrt{5}$.

The *Pearson skewness*, P, of a distribution is defined by

$$P = \frac{3(\mu - M)}{\sigma}\,,$$

where M is the median. Find P for the above distribution and show that $D > P > \gamma$.