Section A: Pure Mathematics

1 **(i)** Find the general solution of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} - \left(\frac{x+2}{x+1}\right) u = 0\,.$$

(ii) Show that substituting $y = z\mathrm{e}^{-x}$ (where z is a function of x) into the second order differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0 \qquad (*)$$

leads to a first order differential equation for $\dfrac{\mathrm{d}z}{\mathrm{d}x}$. Find z and hence show that the general solution of $(*)$ is

$$y = Ax + B\mathrm{e}^{-x}\,,$$

where A and B are arbitrary constants.

(iii) Find the general solution of the differential equation

$$(x+1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = (x+1)^2.$$

2 The polynomial $\mathrm{f}(x)$ is defined by

$$\mathrm{f}(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0\,,$$

where $n \geqslant 2$ and the coefficients $a_0, \ldots, a_{n-1}$ are integers, with $a_0 \neq 0$. Suppose that the equation $\mathrm{f}(x) = 0$ has a rational root p/q, where p and q are integers with no common factor greater than 1, and $q > 0$. By considering $q^{n-1}\mathrm{f}(p/q)$, find the value of q and deduce that any rational root of the equation $\mathrm{f}(x) = 0$ must be an integer.

(i) Show that the nth root of 2 is irrational for $n \geqslant 2$.

(ii) Show that the cubic equation

$$x^3 - x + 1 = 0$$

has no rational roots.

(iii) Show that the polynomial equation

$$x^n - 5x + 7 = 0$$

has no rational roots for $n \geqslant 2$.

3 Show that, provided $q^2 \neq 4p^3$, the polynomial

$$x^3 - 3px + q \qquad\qquad (p \neq 0,\ q \neq 0)$$

can be written in the form

$$a(x-\alpha)^3 + b(x-\beta)^3\,,$$

where α and β are the roots of the quadratic equation $pt^2 - qt + p^2 = 0$, and a and b are constants which you should express in terms of α and β.

Hence show that one solution of the equation $x^3 - 24x + 48 = 0$ is

$$x = \frac{2(2 - 2^{\frac{1}{3}})}{1 - 2^{\frac{1}{3}}}$$

and obtain similar expressions for the other two solutions in terms of ω, where $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$.

Find also the roots of $x^3 - 3px + q = 0$ when $p = r^2$ and $q = 2r^3$ for some non-zero constant r.

4 The following result applies to any function f which is continuous, has positive gradient and satisfies $\mathrm{f}(0) = 0$:

$$ab \leqslant \int_0^a \mathrm{f}(x)\,\mathrm{d}x + \int_0^b \mathrm{f}^{-1}(y)\,\mathrm{d}y\,, \qquad (*)$$

where f^{-1} denotes the inverse function of f, and $a \geqslant 0$ and $b \geqslant 0$.

(i) By considering the graph of $y = \mathrm{f}(x)$, explain briefly why the inequality $(*)$ holds.

In the case $a > 0$ and $b > 0$, state a condition on a and b under which equality holds.

(ii) By taking $\mathrm{f}(x) = x^{p-1}$ in $(*)$, where $p > 1$, show that if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Verify that equality holds under the condition you stated above.

(iii) Show that, for $0 \leqslant a \leqslant \frac{1}{2}\pi$ and $0 \leqslant b \leqslant 1$,

$$ab \leqslant b \arcsin b + \sqrt{1-b^2} - \cos a\,.$$

Deduce that, for $t \geqslant 1$,

$$\arcsin(t^{-1}) \geqslant t - \sqrt{t^2 - 1}\,.$$

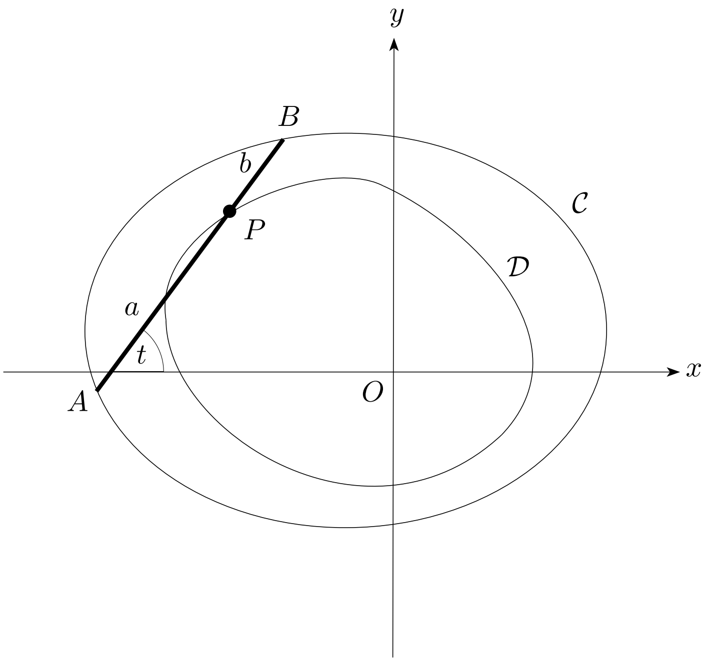
5 A movable point P has cartesian coordinates (x, y), where x and y are functions of t. The polar coordinates of P with respect to the origin O are r and θ. Starting with the expression

$$\tfrac{1}{2}\int r^2 \, \mathrm{d}\theta$$

for the area swept out by OP, obtain the equivalent expression

$$\tfrac{1}{2}\int \left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t} \right) \mathrm{d}t \, . \qquad (*)$$

The ends of a thin straight rod AB lie on a closed convex curve $\mathcal{C}$. The point P on the rod is a fixed distance a from A and a fixed distance b from B. The angle between AB and the positive x direction is t. As A and B move anticlockwise round $\mathcal{C}$, the angle t increases from 0 to 2π and P traces a closed convex curve $\mathcal{D}$ inside $\mathcal{C}$, with the origin O lying inside $\mathcal{D}$, as shown in the diagram.



Let (x, y) be the coordinates of P. Write down the coordinates of A and B in terms of a, b, x, y and t.

The areas swept out by OA, OB and OP are denoted by $[A]$, $[B]$ and $[P]$, respectively. Show, using $(*)$, that

$$[A] = [P] + \pi a^2 - af$$

where

$$f = \tfrac{1}{2}\int_0^{2\pi} \left(\left(x + \frac{\mathrm{d}y}{\mathrm{d}t} \right) \cos t + \left(y - \frac{\mathrm{d}x}{\mathrm{d}t} \right) \sin t \right) \mathrm{d}t \, .$$

Obtain a corresponding expression for $[B]$ involving b. Hence show that the area between the curves $\mathcal{C}$ and $\mathcal{D}$ is πab.

6 The definite integrals T, U, V and X are defined by

$$T = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\operatorname{artanh} t}{t}\,\mathrm{d}t\,, \qquad U = \int_{\ln 2}^{\ln 3} \frac{u}{2\sinh u}\,\mathrm{d}u\,,$$

$$V = -\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{\ln v}{1-v^2}\,\mathrm{d}v\,, \qquad X = \int_{\frac{1}{2}\ln 2}^{\frac{1}{2}\ln 3} \ln(\coth x)\,\mathrm{d}x\,.$$

Show, without evaluating any of them, that T, U, V and X are all equal.

7 Let

$$T_n = \left(\sqrt{a+1} + \sqrt{a}\right)^n \,,$$

where n is a positive integer and a is any given positive integer.

(i) In the case when n is even, show by induction that T_n can be written in the form

$$A_n + B_n\sqrt{a(a+1)}\,,$$

where A_n and B_n are integers (depending on a and n) and $A_n^2 = a(a+1)B_n^2 + 1$.

(ii) In the case when n is odd, show by considering $(\sqrt{a+1} + \sqrt{a})T_m$ where m is even, or otherwise, that T_n can be written in the form

$$C_n\sqrt{a+1} + D_n\sqrt{a}\,,$$

where C_n and D_n are integers (depending on a and n) and $(a+1)C_n^2 = aD_n^2 + 1$.

(iii) Deduce that, for each n, T_n can be written as the sum of the square roots of two consecutive integers.

8 The complex numbers z and w are related by

$$w = \frac{1+\mathrm{i}z}{\mathrm{i}+z}\,.$$

Let $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, where x, y, u and v are real. Express u and v in terms of x and y.

(i) By setting $x = \tan(\theta/2)$, or otherwise, show that if the locus of z is the real axis $y = 0$, $-\infty < x < \infty$, then the locus of w is the circle $u^2 + v^2 = 1$ with one point omitted.

(ii) Find the locus of w when the locus of z is the line segment $y = 0$, $-1 < x < 1$.

(iii) Find the locus of w when the locus of z is the line segment $x = 0$, $-1 < y < 1$.

(iv) Find the locus of w when the locus of z is the line $y = 1$, $-\infty < x < \infty$.

Section B: Mechanics

9 Particles P and Q have masses $3m$ and $4m$, respectively. They lie on the outer curved surface of a smooth circular cylinder of radius a which is fixed with its axis horizontal. They are connected by a light inextensible string of length $\frac{1}{2}\pi a$, which passes over the surface of the cylinder. The particles and the string all lie in a vertical plane perpendicular to the axis of the cylinder, and the axis intersects this plane at O. Initially, the particles are in equilibrium.

Equilibrium is slightly disturbed and Q begins to move downwards. Show that while the two particles are still in contact with the cylinder the angle θ between OQ and the vertical satisfies

$$7a\dot{\theta}^2 + 8g\cos\theta + 6g\sin\theta = 10g\,.$$

(i) Given that Q loses contact with the cylinder first, show that it does so when $\theta = \beta$, where β satisfies

$$15\cos\beta + 6\sin\beta = 10.$$

(ii) Show also that while P and Q are still in contact with the cylinder the tension in the string is $\frac{12}{7}mg(\sin\theta + \cos\theta)$.

10 Particles P and Q, each of mass m, lie initially at rest a distance a apart on a smooth horizontal plane. They are connected by a light elastic string of natural length a and modulus of elasticity $\frac{1}{2}ma\omega^2$, where ω is a constant.

Then P receives an impulse which gives it a velocity u directly away from Q. Show that when the string next returns to length a, the particles have travelled a distance $\frac{1}{2}\pi u/\omega$, and find the speed of each particle.

Find also the total time between the impulse and the subsequent collision of the particles.

11 A thin uniform circular disc of radius a and mass m is held in equilibrium in a horizontal plane a distance b below a horizontal ceiling, where $b > 2a$. It is held in this way by n light inextensible vertical strings, each of length b; one end of each string is attached to the edge of the disc and the other end is attached to a point on the ceiling. The strings are equally spaced around the edge of the disc. One of the strings is attached to the point P on the disc which has coordinates $(a, 0, -b)$ with respect to cartesian axes with origin on the ceiling directly above the centre of the disc.

The disc is then rotated through an angle θ (where $\theta < \pi$) about its vertical axis of symmetry and held at rest by a couple acting in the plane of the disc. Show that the string attached to P now makes an angle ϕ with the vertical, where

$$b \sin \phi = 2a \sin \tfrac{1}{2}\theta \,.$$

Show further that the magnitude of the couple is

$$\frac{mga^2 \sin \theta}{\sqrt{b^2 - 4a^2 \sin^2 \frac{1}{2}\theta}} \,.$$

The disc is now released from rest. Show that its angular speed, ω, when the strings are vertical is given by

$$\frac{a^2\omega^2}{4g} = b - \sqrt{b^2 - 4a^2 \sin^2 \tfrac{1}{2}\theta} \,.$$

Section C: Probability and Statistics

12 The random variable N takes positive integer values and has pgf (probability generating function) $\mathrm{G}(t)$. The random variables X_i, where $i = 1, 2, 3, \ldots$, are independently and identically distributed, each with pgf $\mathrm{H}(t)$. The random variables X_i are also independent of N. The random variable Y is defined by

$$Y = \sum_{i=1}^{N} X_i \,.$$

Given that the pgf of Y is $\mathrm{G}(\mathrm{H}(t))$, show that

$$\mathrm{E}(Y) = \mathrm{E}(N)\mathrm{E}(X_i) \quad \text{and} \quad \mathrm{Var}\,(Y) = \mathrm{Var}\,(N)\big(\mathrm{E}(X_i)\big)^2 + \mathrm{E}(N)\mathrm{Var}\,(X_i)\,.$$

A fair coin is tossed until a head occurs. The total number of tosses is N. The coin is then tossed a further N times and the total number of heads in these N tosses is Y. Find in this particular case the pgf of Y, $\mathrm{E}(Y)$, $\mathrm{Var}\,(Y)$ and $\mathrm{P}(Y = r)$.

13 In this question, the notation $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, so for example $\lfloor \pi \rfloor = 3$ and $\lfloor 3 \rfloor = 3$.

(i) A bag contains n balls, of which b are black. A sample of k balls is drawn, one after another, at random *with* replacement. The random variable X denotes the number of black balls in the sample. By considering

$$\frac{\mathrm{P}(X = r+1)}{\mathrm{P}(X = r)}\,,$$

show that, in the case that it is unique, the most probable number of black balls in the sample is

$$\left\lfloor \frac{(k+1)b}{n} \right\rfloor .$$

Under what circumstances is the answer not unique?

(ii) A bag contains n balls, of which b are black. A sample of k balls (where $k \leqslant b$) is drawn, one after another, at random *without* replacement. Find, in the case that it is unique, the most probable number of black balls in the sample.

Under what circumstances is the answer not unique?