Section A: Pure Mathematics

1 The line L has equation $y = c - mx$, with $m > 0$ and $c > 0$. It passes through the point $R(a, b)$ and cuts the axes at the points $P(p, 0)$ and $Q(0, q)$, where a, b, p and q are all positive. Find p and q in terms of a, b and m.

As L varies with R remaining fixed, show that the minimum value of the sum of the distances of P and Q from the origin is $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$, and find in a similar form the minimum distance between P and Q. (You may assume that any stationary values of these distances are minima.)

2 **(i)** Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0\,,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

3　**(i)**　Sketch the curve $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and add to your diagram the tangent to the curve at the origin and the chord joining the origin to the point $(b, \sin b)$, where $0 < b < \frac{1}{2}\pi$.

By considering areas, show that

$$1 - \tfrac{1}{2}b^2 < \cos b < 1 - \tfrac{1}{2}b \sin b \, .$$

(ii)　By considering the curve $y = a^x$, where $a > 1$, show that

$$\frac{2(a-1)}{a+1} < \ln a < -1 + \sqrt{2a-1} \, .$$

[**Hint**: You may wish to write a^x as $\mathrm{e}^{x \ln a}$.]

4　The curve C has equation $xy = \frac{1}{2}$. The tangents to C at the distinct points $P\big(p, \frac{1}{2p}\big)$ and $Q\big(q, \frac{1}{2q}\big)$, where p and q are positive, intersect at T and the normals to C at these points intersect at N. Show that T is the point

$$\left(\frac{2pq}{p+q} \, , \, \frac{1}{p+q} \right).$$

In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

5　Show that

$$\int_0^{\frac{1}{4}\pi} \sin(2x) \ln(\cos x) \, \mathrm{d}x = \frac{1}{4}(\ln 2 - 1) \, ,$$

and that

$$\int_0^{\frac{1}{4}\pi} \cos(2x) \ln(\cos x) \, \mathrm{d}x = \frac{1}{8}(\pi - \ln 4 - 2) \, .$$

Hence evaluate

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \big(\cos(2x) + \sin(2x) \big) \, \ln \big(\cos x + \sin x \big) \, \mathrm{d}x \, .$$

6 A thin circular path with diameter AB is laid on horizontal ground. A vertical flagpole is erected with its base at a point D on the diameter AB. The angles of elevation of the top of the flagpole from A and B are α and β respectively (both are acute). The point C lies on the circular path with DC perpendicular to AB and the angle of elevation of the top of the flagpole from C is ϕ. Show that $\cot\alpha\cot\beta = \cot^2\phi$.

Show that, for any p and q,

$$\cos p\cos q\sin^2\tfrac{1}{2}(p+q) - \sin p\sin q\cos^2\tfrac{1}{2}(p+q) = \tfrac{1}{2}\cos(p+q) - \tfrac{1}{2}\cos(p+q)\cos(p-q).$$

Deduce that, if p and q are positive and $p+q \leqslant \frac{1}{2}\pi$, then

$$\cot p\cot q \geqslant \cot^2\tfrac{1}{2}(p+q)$$

and hence show that $\phi \leqslant \frac{1}{2}(\alpha+\beta)$ when $\alpha+\beta \leqslant \frac{1}{2}\pi$.

7 A sequence of numbers t_0, t_1, t_2, $\ldots$ satisfies

$$t_{n+2} = pt_{n+1} + qt_n \qquad (n \geqslant 0),$$

where p and q are real. Throughout this question, x, y and z are non-zero real numbers.

(i) Show that, if $t_n = x$ for all values of n, then $p+q=1$ and x can be any (non-zero) real number.

(ii) Show that, if $t_{2n} = x$ and $t_{2n+1} = y$ for all values of n, then $q \pm p = 1$. Deduce that either $x = y$ or $x = -y$, unless p and q take certain values that you should identify.

(iii) Show that, if $t_{3n} = x$, $t_{3n+1} = y$ and $t_{3n+2} = z$ for all values of n, then

$$p^3 + q^3 + 3pq - 1 = 0\,.$$

Deduce that either $p+q=1$ or $(p-q)^2 + (p+1)^2 + (q+1)^2 = 0$. Hence show that either $x = y = z$ or $x+y+z = 0$.

8 **(i)** Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \qquad\qquad (x \neq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0\,.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C\,,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 5y = 0 \qquad\qquad (x \neq 0).$$

Section B: Mechanics

9 A tall shot-putter projects a small shot from a point 2.5 m above the ground, which is horizontal. The speed of projection is $10\,\text{m}\,\text{s}^{-1}$ and the angle of projection is θ above the horizontal. Taking the acceleration due to gravity to be $10\,\text{m}\,\text{s}^{-2}$, show that the time, in seconds, that elapses before the shot hits the ground is

$$\frac{1}{\sqrt{2}}\left(\sqrt{1-c}+\sqrt{2-c}\right),$$

where $c = \cos 2\theta$.

Find an expression for the range in terms of c and show that it is greatest when $c = \frac{1}{5}$.

Show that the extra distance attained by projecting the shot at this angle rather than at an angle of $45°$ is $5(\sqrt{6}-\sqrt{2}-1)$ m.

10 I stand at the top of a vertical well. The depth of the well, from the top to the surface of the water, is D. I drop a stone from the top of the well and measure the time that elapses between the release of the stone and the moment when I hear the splash of the stone entering the water.

In order to gauge the depth of the well, I climb a distance δ down into the well and drop a stone from my new position. The time until I hear the splash is t less than the previous time. Show that

$$t = \sqrt{\frac{2D}{g}} - \sqrt{\frac{2(D-\delta)}{g}} + \frac{\delta}{u},$$

where u is the (constant) speed of sound. Hence show that

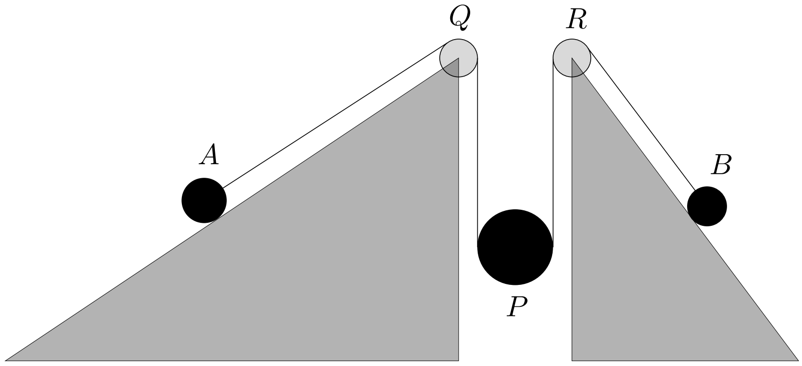
$$D = \tfrac{1}{2}gT^2,$$

where $T = \dfrac{1}{2}\beta + \dfrac{\delta}{\beta g}$ and $\beta = t - \dfrac{\delta}{u}$.

Taking $u = 300\,\text{m}\,\text{s}^{-1}$ and $g = 10\,\text{m}\,\text{s}^{-2}$, show that if $t = \frac{1}{5}$ s and $\delta = 10$ m, the well is approximately 185 m deep.

11 The diagram shows two particles, A of mass $5m$ and B of mass $3m$, connected by a light inextensible string which passes over two smooth, light, fixed pulleys, Q and R, and under a smooth pulley P which has mass M and is free to move vertically.

Particles A and B lie on fixed rough planes inclined to the horizontal at angles of $\arctan \frac{7}{24}$ and $\arctan \frac{4}{3}$ respectively. The segments AQ and RB of the string are parallel to their respective planes, and segments QP and PR are vertical. The coefficient of friction between each particle and its plane is μ.



(i) Given that the system is in equilibrium, with both A and B on the point of moving up their planes, determine the value of μ and show that $M = 6m$.

(ii) In the case when $M = 9m$, determine the initial accelerations of A, B and P in terms of g.

Section C: Probability and Statistics

12 Fire extinguishers may become faulty at any time after manufacture and are tested annually on the anniversary of manufacture.

The time T years after manufacture until a fire extinguisher becomes faulty is modelled by the continuous probability density function

$$\mathrm{f}(t) = \begin{cases} \dfrac{2t}{(1+t^2)^2} & \text{for } t \geqslant 0\,, \\ 0 & \text{otherwise.} \end{cases}$$

A faulty fire extinguisher will fail an annual test with probability p, in which case it is destroyed immediately. A non-faulty fire extinguisher will always pass the test. All of the annual tests are independent.

Show that the probability that a randomly chosen fire extinguisher will be destroyed exactly three years after its manufacture is $p(5p^2 - 13p + 9)/10$.

Find the probability that a randomly chosen fire extinguisher that was destroyed exactly three years after its manufacture was faulty 18 months after its manufacture.

13 I choose at random an integer in the range 10000 to 99999, all choices being equally likely. Given that my choice does not contain the digits 0, 6, 7, 8 or 9, show that the expected number of different digits in my choice is 3.3616.