Section A: Pure Mathematics

1 Write down the general term in the expansion in powers of x of $(1-x^6)^{-2}$.

(i) Find the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}.$$

Obtain also, and simplify, formulae for the coefficient of x^n in the different cases that arise.

(ii) Show that the coefficient of x^{24} in the expansion in powers of x of

$$(1-x^6)^{-2}(1-x^3)^{-1}(1-x)^{-1}$$

is 55, and find the coefficients of x^{25} and x^{66}.

2 If $\mathrm{p}(x)$ and $\mathrm{q}(x)$ are polynomials of degree m and n, respectively, what is the degree of $\mathrm{p}(\mathrm{q}(x))$?

(i) The polynomial $\mathrm{p}(x)$ satisfies

$$\mathrm{p}(\mathrm{p}(\mathrm{p}(x))) - 3\mathrm{p}(x) = -2x$$

for all x. Explain carefully why $\mathrm{p}(x)$ must be of degree 1, and find all polynomials that satisfy this equation.

(ii) Find all polynomials that satisfy

$$2\mathrm{p}(\mathrm{p}(x)) + 3[\mathrm{p}(x)]^2 - 4\mathrm{p}(x) = x^4$$

for all x.

3 Show that, for any function f (for which the integrals exist),

$$\int_0^\infty \mathrm{f}\big(x+\sqrt{1+x^2}\big)\,\mathrm{d}x = \frac{1}{2}\int_1^\infty \left(1+\frac{1}{t^2}\right)\mathrm{f}(t)\,\mathrm{d}t\,.$$

Hence evaluate

$$\int_0^\infty \frac{1}{2x^2+1+2x\sqrt{x^2+1}}\;\mathrm{d}x\,,$$

and, using the substitution $x = \tan\theta$,

$$\int_0^{\frac{1}{2}\pi} \frac{1}{(1+\sin\theta)^3}\,\mathrm{d}\theta\,.$$

4 In this question, you may assume that the infinite series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$

is valid for $|x| < 1$.

(i) Let n be an integer greater than 1. Show that, for any positive integer k,

$$\frac{1}{(k+1)n^{k+1}} < \frac{1}{kn^k}\,.$$

Hence show that $\ln\left(1+\dfrac{1}{n}\right) < \dfrac{1}{n}$. Deduce that

$$\left(1+\frac{1}{n}\right)^n < \mathrm{e}\,.$$

(ii) Show, using an expansion in powers of $\dfrac{1}{y}$, that $\ln\left(\dfrac{2y+1}{2y-1}\right) > \dfrac{1}{y}$ for $y > \frac{1}{2}$.

Deduce that, for any positive integer n,

$$\mathrm{e} < \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}\,.$$

(iii) Use parts (i) and (ii) to show that as $n \to \infty$

$$\left(1+\frac{1}{n}\right)^n \to \mathrm{e}\,.$$

5 **(i)** Sketch the curve $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \frac{1}{(x-a)^2 - 1} \qquad (x \neq a \pm 1),$$

and a is a constant.

(ii) The function $\mathrm{g}(x)$ is defined by

$$\mathrm{g}(x) = \frac{1}{\big((x-a)^2 - 1\big)\big((x-b)^2 - 1\big)} \qquad (x \neq a \pm 1,\ x \neq b \pm 1),$$

where a and b are constants, and $b > a$. Sketch the curves $y = \mathrm{g}(x)$ in the two cases $b > a + 2$ and $b = a + 2$, finding the values of x at the stationary points.

6 A cyclic quadrilateral $ABCD$ has sides AB, BC, CD and DA of lengths a, b, c and d, respectively. The area of the quadrilateral is Q, and angle DAB is θ.

Find an expression for $\cos\theta$ in terms of a, b, c and d, and an expression for $\sin\theta$ in terms of a, b, c, d and Q. Hence show that

$$16Q^2 = 4(ad + bc)^2 - (a^2 + d^2 - b^2 - c^2)^2\,,$$

and deduce that

$$Q^2 = (s-a)(s-b)(s-c)(s-d)\,,$$

where $s = \frac{1}{2}(a + b + c + d)$.

Deduce a formula for the area of a triangle with sides of length a, b and c.

7 Three distinct points, X_1, X_2 and X_3, with position vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ respectively, lie on a circle of radius 1 with its centre at the origin O. The point G has position vector $\frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)$. The line through X_1 and G meets the circle again at the point Y_1 and the points Y_2 and Y_3 are defined correspondingly.

Given that $\overrightarrow{GY_1} = -\lambda_1 \overrightarrow{GX_1}$, where λ_1 is a positive scalar, show that

$$\overrightarrow{OY_1} = \tfrac{1}{3}\big((1 - 2\lambda_1)\mathbf{x}_1 + (1 + \lambda_1)(\mathbf{x}_2 + \mathbf{x}_3)\big)$$

and hence that

$$\lambda_1 = \frac{3 - \alpha - \beta - \gamma}{3 + \alpha - 2\beta - 2\gamma}\,,$$

where $\alpha = \mathbf{x}_2 \,.\, \mathbf{x}_3$, $\beta = \mathbf{x}_3 \,.\, \mathbf{x}_1$ and $\gamma = \mathbf{x}_1 \,.\, \mathbf{x}_2$.

Deduce that $\dfrac{GX_1}{GY_1} + \dfrac{GX_2}{GY_2} + \dfrac{GX_3}{GY_3} = 3\,.$

8 The positive numbers α, β and q satisfy $\beta - \alpha > q$. Show that

$$\frac{\alpha^2 + \beta^2 - q^2}{\alpha\beta} - 2 > 0\,.$$

The sequence u_0, u_1, $\ldots$ is defined by $u_0 = \alpha$, $u_1 = \beta$ and

$$u_{n+1} = \frac{u_n^2 - q^2}{u_{n-1}} \qquad\qquad (n \geqslant 1),$$

where α, β and q are given positive numbers (and α and β are such that no term in the sequence is zero). Prove that $u_n(u_n + u_{n+2}) = u_{n+1}(u_{n-1} + u_{n+1})$. Prove also that

$$u_{n+1} - pu_n + u_{n-1} = 0$$

for some number p which you should express in terms of α, β and q.

Hence, or otherwise, show that if $\beta > \alpha + q$, the sequence is strictly increasing (that is, $u_{n+1} - u_n > 0$ for all n). Comment on the case $\beta = \alpha + q$.

Section B: Mechanics

9 A tennis ball is projected from a height of $2h$ above horizontal ground with speed u and at an angle of α below the horizontal. It travels in a plane perpendicular to a vertical net of height h which is a horizontal distance of a from the point of projection. Given that the ball passes over the net, show that

$$\frac{1}{u^2} < \frac{2(h - a\tan\alpha)}{ga^2\sec^2\alpha}\,.$$

The ball lands before it has travelled a horizontal distance of b from the point of projection. Show that

$$\sqrt{u^2\sin^2\alpha + 4gh} < \frac{bg}{u\cos\alpha} + u\sin\alpha\,.$$

Hence show that

$$\tan\alpha < \frac{h(b^2 - 2a^2)}{ab(b-a)}\,.$$

10 A hollow circular cylinder of internal radius r is held fixed with its axis horizontal. A uniform rod of length $2a$ (where $a < r$) rests in equilibrium inside the cylinder inclined at an angle of θ to the horizontal, where $\theta \neq 0$. The vertical plane containing the rod is perpendicular to the axis of the cylinder. The coefficient of friction between the cylinder and each end of the rod is μ, where $\mu > 0$.

Show that, if the rod is on the point of slipping, then the normal reactions R_1 and R_2 of the lower and higher ends of the rod, respectively, on the cylinder are related by

$$\mu(R_1 + R_2) = (R_1 - R_2)\tan\phi$$

where ϕ is the angle between the rod and the radius to an end of the rod.

Show further that

$$\tan\theta = \frac{\mu r^2}{r^2 - a^2(1+\mu^2)}\,.$$

Deduce that $\lambda < \phi$, where $\tan\lambda = \mu$.

11 A small block of mass km is initially at rest on a smooth horizontal surface. Particles P_1, P_2, P_3, ... are fired, in order, along the surface from a fixed point towards the block. The mass of the ith particle is im (i = 1, 2, ...) and the speed at which it is fired is u/i. Each particle that collides with the block is embedded in it. Show that, if the nth particle collides with the block, the speed of the block after the collision is

$$\frac{2nu}{2k + n(n+1)}\,.$$

In the case $2k = N(N+1)$, where N is a positive integer, determine the number of collisions that occur. Show that the total kinetic energy lost in all the collisions is

$$\tfrac{1}{2}mu^2\left(\sum_{n=2}^{N+1}\frac{1}{n}\right).$$

Section C: Probability and Statistics

12 A modern villa has complicated lighting controls. In order for the light in the swimming pool to be on, a particular switch in the hallway must be on and a particular switch in the kitchen must be on. There are four identical switches in the hallway and four identical switches in the kitchen. Guests cannot tell whether the switches are on or off, or what they control.

Each Monday morning a guest arrives, and the switches in the hallway are either all on or all off. The probability that they are all on is p and the probability that they are all off is $1-p$. The switches in the kitchen are each on or off, independently, with probability $\frac{1}{2}$.

(i) On the first Monday, a guest presses one switch in the hallway at random and one switch in the kitchen at random. Find the probability that the swimming pool light is on at the end of this process. Show that the probability that the guest has pressed the swimming pool light switch in the hallway, given that the light is on at the end of the process, is $\dfrac{1-p}{1+2p}$.

(ii) On each of seven Mondays, guests go through the above process independently of each other, and each time the swimming pool light is found to be on at the end of the process. Given that the most likely number of days on which the swimming pool light switch in the hallway was pressed is 3, show that $\frac{1}{4} < p < \frac{5}{14}$.

13 In this question, you may assume that $\displaystyle\int_0^\infty \mathrm{e}^{-x^2/2}\,\mathrm{d}x = \sqrt{\tfrac{1}{2}\pi}$.

The number of supermarkets situated in any given region can be modelled by a Poisson random variable, where the mean is k times the area of the given region. Find the probability that there are no supermarkets within a circle of radius y.

The random variable Y denotes the distance between a randomly chosen point in the region and the nearest supermarket. Write down $\mathrm{P}(Y < y)$ and hence show that the probability density function of Y is $2\pi yk\mathrm{e}^{-\pi ky^2}$ for $y \geqslant 0$.

Find $\mathrm{E}(Y)$ and show that $\mathrm{Var}\,(Y) = \dfrac{4-\pi}{4\pi k}$.