

**Section A: Pure Mathematics**

1    Given that  $z = y^n \left( \frac{dy}{dx} \right)^2$ , show that

$$\frac{dz}{dx} = y^{n-1} \frac{dy}{dx} \left( n \left( \frac{dy}{dx} \right)^2 + 2y \frac{d^2y}{dx^2} \right).$$

(i)    Use the above result to show that the solution to the equation

$$\left( \frac{dy}{dx} \right)^2 + 2y \frac{d^2y}{dx^2} = \sqrt{y} \quad (y > 0)$$

that satisfies  $y = 1$  and  $\frac{dy}{dx} = 0$  when  $x = 0$  is  $y = \left( \frac{3}{8}x^2 + 1 \right)^{\frac{2}{3}}$ .

(ii)    Find the solution to the equation

$$\left( \frac{dy}{dx} \right)^2 - y \frac{d^2y}{dx^2} + y^2 = 0$$

that satisfies  $y = 1$  and  $\frac{dy}{dx} = 0$  when  $x = 0$ .

2    In this question,  $|x| < 1$  and you may ignore issues of convergence.

(i)    Simplify

$$(1-x)(1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^n}),$$

where  $n$  is a positive integer, and deduce that

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4) \cdots (1+x^{2^n}) + \frac{x^{2^{n+1}}}{1-x}.$$

Deduce further that

$$\ln(1-x) = - \sum_{r=0}^{\infty} \ln(1+x^{2^r}),$$

and hence that

$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \cdots.$$

(ii)    Show that

$$\frac{1+2x}{1+x+x^2} = \frac{1-2x}{1-x+x^2} + \frac{2x-4x^3}{1-x^2+x^4} + \frac{4x^3-8x^7}{1-x^4+x^8} + \cdots.$$

3 It is given that the two curves

$$y = 4 - x^2 \quad \text{and} \quad mx = k - y^2,$$

where  $m > 0$ , touch exactly once.

(i) In each of the following four cases, sketch the two curves on a single diagram, noting the coordinates of any intersections with the axes:

(a)  $k < 0$ ;

(b)  $0 < k < 16, k/m < 2$ ;

(c)  $k > 16, k/m > 2$ ;

(d)  $k > 16, k/m < 2$ .

(ii) Now set  $m = 12$ .

Show that the  $x$ -coordinate of any point at which the two curves meet satisfies

$$x^4 - 8x^2 + 12x + 16 - k = 0.$$

Let  $a$  be the value of  $x$  at the point where the curves touch. Show that  $a$  satisfies

$$a^3 - 4a + 3 = 0$$

and hence find the three possible values of  $a$ .

Derive also the equation

$$k = -4a^2 + 9a + 16.$$

Which of the four sketches in part (i) arise?

4 (i) Show that

$$\sum_{n=1}^{\infty} \frac{n+1}{n!} = 2e - 1$$

and

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!} = 5e - 1.$$

Sum the series  $\sum_{n=1}^{\infty} \frac{(2n-1)^3}{n!}$ .

(ii) Sum the series  $\sum_{n=0}^{\infty} \frac{(n^2+1)2^{-n}}{(n+1)(n+2)}$ , giving your answer in terms of natural logarithms.

- 5 (i) The point with coordinates  $(a, b)$ , where  $a$  and  $b$  are rational numbers, is called:  
 an *integer rational point* if both  $a$  and  $b$  are integers;  
 a *non-integer rational point* if neither  $a$  nor  $b$  is an integer.
- (a) Write down an integer rational point and a non-integer rational point on the circle  $x^2 + y^2 = 1$ .
- (b) Write down an integer rational point on the circle  $x^2 + y^2 = 2$ . Simplify

$$(\cos \theta + \sqrt{m} \sin \theta)^2 + (\sin \theta - \sqrt{m} \cos \theta)^2$$

and hence obtain a non-integer rational point on the circle  $x^2 + y^2 = 2$ .

- (ii) The point with coordinates  $(p + \sqrt{2}q, r + \sqrt{2}s)$ , where  $p, q, r$  and  $s$  are rational numbers, is called:  
 an *integer 2-rational point* if all of  $p, q, r$  and  $s$  are integers;  
 a *non-integer 2-rational point* if none of  $p, q, r$  and  $s$  is an integer.
- (a) Write down an integer 2-rational point, and obtain a non-integer 2-rational point, on the circle  $x^2 + y^2 = 3$ .
- (b) Obtain a non-integer 2-rational point on the circle  $x^2 + y^2 = 11$ .
- (c) Obtain a non-integer 2-rational point on the hyperbola  $x^2 - y^2 = 7$ .

- 6 Let  $x + iy$  be a root of the quadratic equation  $z^2 + pz + 1 = 0$ , where  $p$  is a real number. Show that  $x^2 - y^2 + px + 1 = 0$  and  $(2x + p)y = 0$ . Show further that

$$\text{either } p = -2x \text{ or } p = -(x^2 + 1)/x \text{ with } x \neq 0.$$

Hence show that the set of points in the Argand diagram that can (as  $p$  varies) represent roots of the quadratic equation consists of the real axis with one point missing and a circle. This set of points is called the *root locus* of the quadratic equation.

Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + z + 1 = 0$$

and the root locus of the equation

$$pz^2 + p^2z + 2 = 0.$$

- 7** A pain-killing drug is injected into the bloodstream. It then diffuses into the brain, where it is absorbed. The quantities at time  $t$  of the drug in the blood and the brain respectively are  $y(t)$  and  $z(t)$ . These satisfy

$$\dot{y} = -2(y - z), \quad \dot{z} = -\dot{y} - 3z,$$

where the dot denotes differentiation with respect to  $t$ .

Obtain a second order differential equation for  $y$  and hence derive the solution

$$y = Ae^{-t} + Be^{-6t}, \quad z = \frac{1}{2}Ae^{-t} - 2Be^{-6t},$$

where  $A$  and  $B$  are arbitrary constants.

- (i) Obtain the solution that satisfies  $z(0) = 0$  and  $y(0) = 5$ . The quantity of the drug in the brain for this solution is denoted by  $z_1(t)$ .
- (ii) Obtain the solution that satisfies  $z(0) = z(1) = c$ , where  $c$  is a given constant. The quantity of the drug in the brain for this solution is denoted by  $z_2(t)$ .
- (iii) Show that for  $0 \leq t \leq 1$ ,

$$z_2(t) = \sum_{n=-\infty}^0 z_1(t-n),$$

provided  $c$  takes a particular value that you should find.

- 8** The sequence  $F_0, F_1, F_2, \dots$  is defined by  $F_0 = 0, F_1 = 1$  and, for  $n \geq 0$ ,

$$F_{n+2} = F_{n+1} + F_n.$$

- (i) Show that  $F_0F_3 - F_1F_2 = F_2F_5 - F_3F_4$ .
- (ii) Find the values of  $F_nF_{n+3} - F_{n+1}F_{n+2}$  in the two cases that arise.
- (iii) Prove that, for  $r = 1, 2, 3, \dots$ ,

$$\arctan\left(\frac{1}{F_{2r}}\right) = \arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right)$$

and hence evaluate the following sum (which you may assume converges):

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right).$$

## Section B: Mechanics

- 9** A pulley consists of a disc of radius  $r$  with centre  $O$  and a light thin axle through  $O$  perpendicular to the plane of the disc. The disc is non-uniform, its mass is  $M$  and its centre of mass is at  $O$ . The axle is fixed and horizontal.

Two particles, of masses  $m_1$  and  $m_2$  where  $m_1 > m_2$ , are connected by a light inextensible string which passes over the pulley. The contact between the string and the pulley is rough enough to prevent the string sliding. The pulley turns and the vertical force on the axle is found, by measurement, to be  $P + Mg$ .

- (i) The moment of inertia of the pulley about its axle is calculated assuming that the pulley rotates without friction about its axle. Show that the calculated value is

$$\frac{((m_1 + m_2)P - 4m_1m_2g)r^2}{(m_1 + m_2)g - P}. \quad (*)$$

- (ii) Instead, the moment of inertia of the pulley about its axle is calculated assuming that a couple of magnitude  $C$  due to friction acts on the axle of the pulley. Determine whether this calculated value is greater or smaller than (\*).

Show that  $C < (m_1 - m_2)rg$ .

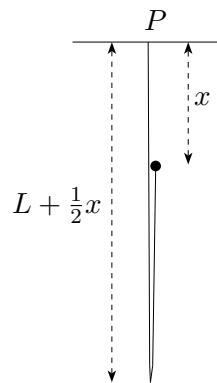
- 10** A small ring of mass  $m$  is free to slide without friction on a hoop of radius  $a$ . The hoop is fixed in a vertical plane. The ring is connected by a light elastic string of natural length  $a$  to the highest point of the hoop. The ring is initially at rest at the lowest point of the hoop and is then slightly displaced. In the subsequent motion the angle of the string to the downward vertical is  $\phi$ . Given that the ring first comes to rest just as the string becomes slack, find an expression for the modulus of elasticity of the string in terms of  $m$  and  $g$ .

Show that, throughout the motion, the magnitude  $R$  of the reaction between the ring and the hoop is given by

$$R = (12 \cos^2 \phi - 15 \cos \phi + 5)mg$$

and that  $R$  is non-zero throughout the motion.

- 11 One end of a thin heavy uniform inextensible perfectly flexible rope of length  $2L$  and mass  $2M$  is attached to a fixed point  $P$ . A particle of mass  $m$  is attached to the other end. Initially, the particle is held at  $P$  and the rope hangs vertically in a loop below  $P$ . The particle is then released so that it and a section of the rope (of decreasing length) fall vertically as shown in the diagram.



You may assume that each point on the moving section of the rope falls at the same speed as the particle. Given that energy is conserved, show that, when the particle has fallen a distance  $x$  (where  $x < 2L$ ), its speed  $v$  is given by

$$v^2 = \frac{2gx(mL + ML - \frac{1}{4}Mx)}{mL + ML - \frac{1}{2}Mx}.$$

Hence show that the acceleration of the particle is

$$g + \frac{Mgx(mL + ML - \frac{1}{4}Mx)}{2(mL + ML - \frac{1}{2}Mx)^2}.$$

Deduce that the acceleration of the particle after it is released is greater than  $g$ .

## Section C: Probability and Statistics

- 12 (i) A point  $P$  lies in an equilateral triangle  $ABC$  of height 1. The perpendicular distances from  $P$  to the sides  $AB$ ,  $BC$  and  $CA$  are  $x_1$ ,  $x_2$  and  $x_3$ , respectively. By considering the areas of triangles with one vertex at  $P$ , show that  $x_1 + x_2 + x_3 = 1$ .

Suppose now that  $P$  is placed at random in the equilateral triangle (so that the probability of it lying in any given region of the triangle is proportional to the area of that region). The perpendicular distances from  $P$  to the sides  $AB$ ,  $BC$  and  $CA$  are random variables  $X_1$ ,  $X_2$  and  $X_3$ , respectively. In the case  $X_1 = \min(X_1, X_2, X_3)$ , give a sketch showing the region of the triangle in which  $P$  lies.

Let  $X = \min(X_1, X_2, X_3)$ . Show that the probability density function for  $X$  is given by

$$f(x) = \begin{cases} 6(1 - 3x) & 0 \leq x \leq \frac{1}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the expected value of  $X$ .

- (ii) A point is chosen at random in a regular tetrahedron of height 1. Find the expected value of the distance from the point to the closest face.  
[The volume of a tetrahedron is  $\frac{1}{3} \times \text{area of base} \times \text{height}$  and its centroid is a distance  $\frac{1}{4} \times \text{height}$  from the base.]

- 13 (i) The random variable  $Z$  has a Normal distribution with mean 0 and variance 1. Show that the expectation of  $Z$  given that  $a < Z < b$  is

$$\frac{\exp(-\frac{1}{2}a^2) - \exp(-\frac{1}{2}b^2)}{\sqrt{2\pi} (\Phi(b) - \Phi(a))},$$

where  $\Phi$  denotes the cumulative distribution function for  $Z$ .

- (ii) The random variable  $X$  has a Normal distribution with mean  $\mu$  and variance  $\sigma^2$ . Show that

$$E(X | X > 0) = \mu + \sigma E(Z | Z > -\mu/\sigma).$$

Hence, or otherwise, show that the expectation,  $m$ , of  $|X|$  is given by

$$m = \mu(1 - 2\Phi(-\mu/\sigma)) + \sigma\sqrt{2/\pi} \exp(-\frac{1}{2}\mu^2/\sigma^2).$$

Obtain an expression for the variance of  $|X|$  in terms of  $\mu$ ,  $\sigma$  and  $m$ .