Section A: Pure Mathematics

1 **(i)** Use the substitution $\sqrt{x} = y$ (where $y \geqslant 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0\,.$$

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2}\; - 22 = 0\,;$

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3}\; - 9 = 0\,.$

2 In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that $\lfloor 2.9 \rfloor = 2 = \lfloor 2.0 \rfloor$ and $\lfloor -1.5 \rfloor = -2$.

The function f is defined, for $x \neq 0$, by $\mathrm{f}(x) = \dfrac{\lfloor x \rfloor}{x}$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \leqslant x \leqslant 3$ (with $x \neq 0$).

(ii) By considering the line $y = \frac{7}{12}$ on your graph, or otherwise, solve the equation $\mathrm{f}(x) = \frac{7}{12}$.

Solve also the equations $\mathrm{f}(x) = \frac{17}{24}$ and $\mathrm{f}(x) = \frac{4}{3}$.

(iii) Find the largest root of the equation $\mathrm{f}(x) = \frac{9}{10}$.

Give necessary and sufficient conditions, in the form of inequalities, for the equation $\mathrm{f}(x) = c$ to have exactly n roots, where $n \geqslant 1$.

3 For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points $P_1, P_2, \ldots$ are defined by $P_1 = X * Y$ and, for $n \geqslant 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

4 **(i)** Show that, for $n > 0$,

$$\int_0^{\frac{1}{4}\pi} \tan^n x \, \sec^2 x \, \mathrm{d}x = \frac{1}{n+1} \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} \sec^n x \, \tan x \, \mathrm{d}x = \frac{(\sqrt{2})^n - 1}{n}.$$

(ii) Evaluate the following integrals:

$$\int_0^{\frac{1}{4}\pi} x \, \sec^4 x \, \tan x \, \mathrm{d}x \quad \text{and} \quad \int_0^{\frac{1}{4}\pi} x^2 \sec^2 x \, \tan x \, \mathrm{d}x.$$

5 The point P has coordinates (x, y) which satisfy

$$x^2 + y^2 + kxy + 3x + y = 0\,.$$

(i) Sketch the locus of P in the case $k = 0$, giving the points of intersection with the coordinate axes.

(ii) By factorising $3x^2 + 3y^2 + 10xy$, or otherwise, sketch the locus of P in the case $k = \frac{10}{3}$, giving the points of intersection with the coordinate axes.

(iii) In the case $k = 2$, let Q be the point obtained by rotating P clockwise about the origin by an angle θ, so that the coordinates (X, Y) of Q are given by

$$X = x\cos\theta + y\sin\theta\,, \quad Y = -x\sin\theta + y\cos\theta\,.$$

Show that, for $\theta = 45^\circ$, the locus of Q is $\sqrt{2}Y = (\sqrt{2}X + 1)^2 - 1$.

Hence, or otherwise, sketch the locus of P in the case $k = 2$, giving the equation of the line of symmetry.

6 By considering the coefficient of x^r in the series for $(1 + x)(1 + x)^n$, or otherwise, obtain the following relation between binomial coefficients:

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \qquad (1 \leqslant r \leqslant n).$$

The sequence of numbers B_0, B_1, B_2, $\ldots$ is defined by

$$B_{2m} = \sum_{j=0}^{m} \binom{2m-j}{j} \quad \text{and} \quad B_{2m+1} = \sum_{k=0}^{m} \binom{2m+1-k}{k}.$$

Show that $B_{n+2} - B_{n+1} = B_n$ $(n = 0, 1, 2, \ldots)$.

What is the relation between the sequence B_0, B_1, B_2, $\ldots$ and the Fibonacci sequence F_0, F_1, F_2, $\ldots$ defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$?

7 **(i)** Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

8 **(i)** The functions $\mathrm{a}, \mathrm{b}, \mathrm{c}$ and d are defined by

$$\begin{aligned}
&\mathrm{a}(x) = x^2 \quad (-\infty < x < \infty),\\
&\mathrm{b}(x) = \ln x \quad (x > 0),\\
&\mathrm{c}(x) = 2x \quad (-\infty < x < \infty),\\
&\mathrm{d}(x) = \sqrt{x} \quad (x \geqslant 0)\,.
\end{aligned}$$

Write down the following composite functions, giving the domain and range of each:

$$\mathrm{cb}, \qquad \mathrm{ab}, \qquad \mathrm{da}, \qquad \mathrm{ad}.$$

(ii) The functions f and g are defined by

$$\begin{aligned}
&\mathrm{f}(x) = \sqrt{x^2 - 1} \quad (|x| \geqslant 1),\\
&\mathrm{g}(x) = \sqrt{x^2 + 1} \quad (-\infty < x < \infty).
\end{aligned}$$

Determine the composite functions fg and gf, giving the domain and range of each.

(iii) Sketch the graphs of the functions h and k defined by

$$\begin{aligned}
&\mathrm{h}(x) = x + \sqrt{x^2 - 1} \quad (x \geqslant 1),\\
&\mathrm{k}(x) = x - \sqrt{x^2 - 1} \quad (|x| \geqslant 1),
\end{aligned}$$

justifying the main features of the graphs, and giving the equations of any asymptotes. Determine the domain and range of the composite function kh.

Section B: Mechanics

9 Two particles, A and B, are projected simultaneously towards each other from two points which are a distance d apart in a horizontal plane. Particle A has mass m and is projected at speed u at angle α above the horizontal. Particle B has mass M and is projected at speed v at angle β above the horizontal. The trajectories of the two particles lie in the same vertical plane.

The particles collide directly when each is at its point of greatest height above the plane. Given that both A and B return to their starting points, and that momentum is conserved in the collision, show that

$$m \cot \alpha = M \cot \beta \,.$$

Show further that the collision occurs at a point which is a horizontal distance b from the point of projection of A where

$$b = \frac{Md}{m+M}\,,$$

and find, in terms of b and α, the height above the horizontal plane at which the collision occurs.

10 Two parallel vertical barriers are fixed a distance d apart on horizontal ice. A small ice hockey puck moves on the ice backwards and forwards between the barriers, in the direction perpendicular to the barriers, colliding with each in turn. The coefficient of friction between the puck and the ice is μ and the coefficient of restitution between the puck and each of the barriers is r.

The puck starts at one of the barriers, moving with speed v towards the other barrier. Show that

$$v_{i+1}^2 - r^2 v_i^2 = -2r^2 \mu g d$$

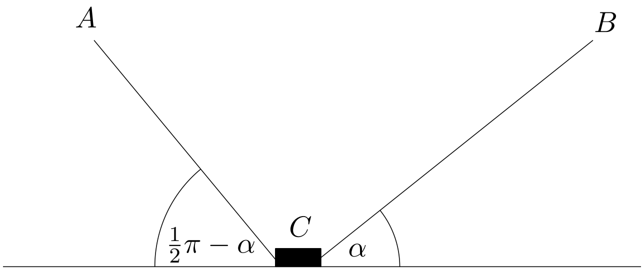
where v_i is the speed of the puck just after its ith collision.

The puck comes to rest against one of the barriers after traversing the gap between them n times. In the case $r \neq 1$, express n in terms of r and k, where $k = \dfrac{v^2}{2\mu g d}$. If $r = \mathrm{e}^{-1}$ (where e is the base of natural logarithms) show that

$$n = \tfrac{1}{2} \ln\big(1 + k(\mathrm{e}^2 - 1)\big)\,.$$

Give an expression for n in the case $r = 1$.

11



The diagram shows a small block C of weight W initially at rest on a rough horizontal surface. The coefficient of friction between the block and the surface is μ. Two light strings, AC and BC, are attached to the block, making angles $\frac{1}{2}\pi - \alpha$ and α to the horizontal, respectively. The tensions in AC and BC are $T \sin\beta$ and $T\cos\beta$ respectively, where $0 < \alpha + \beta < \frac{1}{2}\pi$.

(i) In the case $W > T\sin(\alpha+\beta)$, show that the block will remain at rest provided

$$W \sin\lambda \geqslant T\cos(\alpha + \beta - \lambda)\,,$$

where λ is the acute angle such that $\tan\lambda = \mu$.

(ii) In the case $W = T\tan\phi$, where $2\phi = \alpha + \beta$, show that the block will start to move in a direction that makes an angle ϕ with the horizontal.

Section C: Probability and Statistics

12 Each day, I have to take k different types of medicine, one tablet of each. The tablets are identical in appearance. When I go on holiday for n days, I put n tablets of each type in a container and on each day of the holiday I select k tablets at random from the container.

(i) In the case $k = 3$, show that the probability that I will select one tablet of each type on the first day of a three-day holiday is $\frac{9}{28}$.

Write down the probability that I will be left with one tablet of each type on the last day (irrespective of the tablets I select on the first day).

(ii) In the case $k = 3$, find the probability that I will select one tablet of each type on the first day of an n-day holiday.

(iii) In the case $k = 2$, find the probability that I will select one tablet of each type on each day of an n-day holiday, and use Stirling's approximation

$$n! \approx \sqrt{2n\pi}\left(\frac{n}{\mathrm{e}}\right)^n$$

to show that this probability is approximately $2^{-n}\sqrt{n\pi}$.

13 From the integers $1, 2, \ldots, 52$, I choose seven (distinct) integers at random, all choices being equally likely. From these seven, I discard any pair that sum to 53. Let X be the random variable the value of which is the number of discarded pairs. Find the probability distribution of X and show that $\mathrm{E}(X) = \frac{7}{17}$.

Note: $7 \times 17 \times 47 = 5593$.