Section A: Pure Mathematics

1 Given that $t = \tan \frac{1}{2}x$, show that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}(1+t^2)$ and $\sin x = \dfrac{2t}{1+t^2}$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,\mathrm{d}x = \frac{2}{\sqrt{1-a^2}}\arctan\frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0 < a < 1).$$

Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,\mathrm{d}x \qquad (n \geqslant 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate I_3.

2 In this question, you may ignore questions of convergence.

Let $y = \dfrac{\arcsin x}{\sqrt{1-x^2}}$. Show that

$$(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x} - xy - 1 = 0$$

and prove that, for any positive integer n,

$$(1-x^2)\frac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}} - (2n+3)x\frac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}} - (n+1)^2\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = 0\,.$$

Hence obtain the Maclaurin series for $\dfrac{\arcsin x}{\sqrt{1-x^2}}$, giving the general term for odd and for even powers of x.

Evaluate the infinite sum

$$1 + \frac{1}{3!} + \frac{2^2}{5!} + \frac{2^2\times 3^2}{7!} + \cdots + \frac{2^2\times 3^2\times\cdots\times n^2}{(2n+1)!} + \cdots\,.$$

3 The four vertices P_i ($i = 1, 2, 3, 4$) of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ ($i = 1, 2, 3, 4$). Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \,.\, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i ($i = 1, 2, 3, 4$).

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \,.\, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8\,.$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2\,.$$

By letting the coordinates of X be (x, y, z), show further that $\sum_{i=1}^{4} (XP_i)^4$ is independent of the position of X.

4 Show that $(z - \mathrm{e}^{\mathrm{i}\theta})(z - \mathrm{e}^{-\mathrm{i}\theta}) = z^2 - 2z\cos\theta + 1\,.$

Write down the $(2n)$th roots of -1 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leqslant \pi$, and deduce that

$$z^{2n} + 1 = \prod_{k=1}^{n} \left(z^2 - 2z\cos\left(\tfrac{(2k-1)\pi}{2n}\right) + 1 \right)\,.$$

Here, n is a positive integer, and the $\prod$ notation denotes the product.

(i) By substituting $z = \mathrm{i}$ show that, when n is even,

$$\cos\left(\tfrac{\pi}{2n}\right)\cos\left(\tfrac{3\pi}{2n}\right)\cos\left(\tfrac{5\pi}{2n}\right)\cdots\cos\left(\tfrac{(2n-1)\pi}{2n}\right) = (-1)^{\frac{1}{2}n}2^{1-n}\,.$$

(ii) Show that, when n is odd,

$$\cos^2\left(\tfrac{\pi}{2n}\right)\cos^2\left(\tfrac{3\pi}{2n}\right)\cos^2\left(\tfrac{5\pi}{2n}\right)\cdots\cos^2\left(\tfrac{(n-2)\pi}{2n}\right) = n2^{1-n}\,.$$

You may use without proof the fact that $1 + z^{2n} = (1 + z^2)(1 - z^2 + z^4 - \cdots + z^{2n-2})$ when n is odd.

5 In this question, you may assume that, if a, b and c are positive integers such that a and b are coprime and a divides bc, then a divides c. (Two positive integers are said to be *coprime* if their highest common factor is 1.)

(i) Suppose that there are positive integers p, q, n and N such that p and q are coprime and $q^n N = p^n$. Show that $N = kp^n$ for some positive integer k and deduce the value of q.

Hence prove that, for any positive integers n and N, $\sqrt[n]{N}$ is either a positive integer or irrational.

(ii) Suppose that there are positive integers a, b, c and d such that a and b are coprime and c and d are coprime, and $a^a d^b = b^a c^b$. Prove that $d^b = b^a$ and deduce that, if p is a prime factor of d, then p is also a prime factor of b.

If p^m and p^n are the highest powers of the prime number p that divide d and b, respectively, express b in terms of a, m and n and hence show that $p^n \leqslant n$. Deduce the value of b. (You may assume that if $x > 0$ and $y \geqslant 2$ then $y^x > x$.)

Hence prove that, if r is a positive rational number such that r^r is rational, then r is a positive integer.

6 Let z and w be complex numbers. Use a diagram to show that $|z - w| \leqslant |z| + |w|$.

For any complex numbers z and w, E is defined by

$$E = zw^* + z^*w + 2|zw|\,.$$

(i) Show that $|z - w|^2 = (|z| + |w|)^2 - E$, and deduce that E is real and non-negative.

(ii) Show that $|1 - zw^*|^2 = (1 + |zw|)^2 - E$.

Hence show that, if both $|z| > 1$ and $|w| > 1$, then

$$\frac{|z - w|}{|1 - zw^*|} \leqslant \frac{|z| + |w|}{1 + |zw|}\,.$$

Does this inequality also hold if both $|z| < 1$ and $|w| < 1$?

7 **(i)** Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3 = 0$ with $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \tfrac{1}{2}y^4\,.$$

Show by differentiation that E is constant and deduce that $|y(x)| \leqslant 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + x\dfrac{\mathrm{d}v}{\mathrm{d}x} + \sinh v = 0$ with $v = \ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$, and let

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 + 2\cosh v\,.$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x} \leqslant 0$ for $x \geqslant 0$ and deduce that $\cosh v(x) \leqslant \frac{5}{3}$ for $x \geqslant 0$.

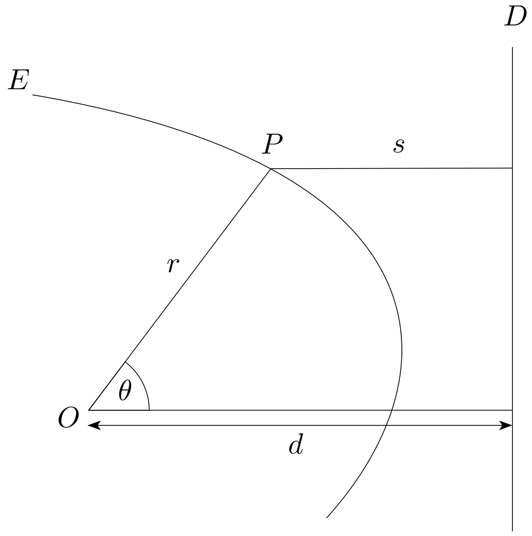
(iii) Let $w(x)$ be a solution of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + (5\cosh x - 4\sinh x - 3)\frac{\mathrm{d}w}{\mathrm{d}x} + (w\cosh w + 2\sinh w) = 0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{1}{\sqrt{2}}$ and $w = 0$ at $x = 0$. Show that $\cosh w(x) \leqslant \frac{5}{4}$ for $x \geqslant 0$.

8 Evaluate $\displaystyle\sum_{r=0}^{n-1} \mathrm{e}^{2\mathrm{i}(\alpha + r\pi/n)}$ where α is a fixed angle and $n \geqslant 2$.

The fixed point O is a distance d from a fixed line D. For any point P, let s be the distance from P to D and let r be the distance from P to O. Write down an expression for s in terms of d, r and the angle θ, where θ is as shown in the diagram below.



The curve E shown in the diagram is such that, for any point P on E, the relation $r = ks$ holds, where k is a fixed number with $0 < k < 1$.

Each of the n lines $L_1, \ldots, L_n$ passes through O and the angle between adjacent lines is $\frac{\pi}{n}$. The line L_j ($j = 1, \ldots, n$) intersects E in two points forming a chord of length l_j. Show that, for $n \geqslant 2$,

$$\sum_{j=1}^{n} \frac{1}{l_j} = \frac{(2-k^2)n}{4kd}\,.$$

Section B: Mechanics

9 A sphere of radius R and uniform density ρ_{s} is floating in a large tank of liquid of uniform density ρ. Given that the centre of the sphere is a distance x above the level of the liquid, where $x < R$, show that the volume of liquid displaced is

$$\frac{\pi}{3}(2R^3 - 3R^2x + x^3)\,.$$

The sphere is acted upon by two forces only: its weight and an upward force equal in magnitude to the weight of the liquid it has displaced. Show that

$$4R^3\rho_{\text{s}}(g + \ddot{x}) = (2R^3 - 3R^2x + x^3)\rho g\,.$$

Given that the sphere is in equilibrium when $x = \frac{1}{2}R$, find ρ_{s} in terms of ρ. Find, in terms of R and g, the period of small oscillations about this equilibrium position.

10 A uniform rod AB has mass M and length $2a$. The point P lies on the rod a distance $a - x$ from A. Show that the moment of inertia of the rod about an axis through P and perpendicular to the rod is

$$\tfrac{1}{3}M(a^2 + 3x^2)\,.$$

The rod is free to rotate, in a horizontal plane, about a fixed vertical axis through P. Initially the rod is at rest. The end B is struck by a particle of mass m moving horizontally with speed u in a direction perpendicular to the rod. The coefficient of restitution between the rod and the particle is e. Show that the angular velocity of the rod immediately after impact is

$$\frac{3mu(1+e)(a+x)}{M(a^2+3x^2) + 3m(a+x)^2}\,.$$

In the case $m = 2M$, find the value of x for which the angular velocity is greatest and show that this angular velocity is $u(1+e)/a$.

11 An equilateral triangle, comprising three light rods each of length $\sqrt{3}a$, has a particle of mass m attached to each of its vertices. The triangle is suspended horizontally from a point vertically above its centre by three identical springs, so that the springs and rods form a tetrahedron. Each spring has natural length a and modulus of elasticity kmg, and is light. Show that when the springs make an angle θ with the horizontal the tension in each spring is

$$\frac{kmg(1 - \cos\theta)}{\cos\theta}\,.$$

Given that the triangle is in equilibrium when $\theta = \frac{1}{6}\pi$, show that $k = 4\sqrt{3} + 6$.

The triangle is released from rest from the position at which $\theta = \frac{1}{3}\pi$. Show that when it passes through the equilibrium position its speed V satisfies

$$V^2 = \frac{4ag}{3}(6 + \sqrt{3})\,.$$

Section C: Probability and Statistics

12 A list consists only of letters A and B arranged in a row. In the list, there are a letter As and b letter Bs, where $a \geqslant 2$ and $b \geqslant 2$, and $a + b = n$. Each possible ordering of the letters is equally probable. The random variable X_1 is defined by

$$X_1 = \begin{cases} 1 & \text{if the first letter in the row is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variables X_k $(2 \leqslant k \leqslant n)$ are defined by

$$X_k = \begin{cases} 1 & \text{if the } (k-1)\text{th letter is } B \text{ and the } k\text{th is } A; \\ 0 & \text{otherwise.} \end{cases}$$

The random variable S is defined by $S = \sum_{i=1}^{n} X_i$.

(i) Find expressions for $\mathrm{E}(X_i)$, distinguishing between the cases $i = 1$ and $i \neq 1$, and show that $\mathrm{E}(S) = \dfrac{a(b+1)}{n}$.

(ii) Show that:

(a) for $j \geqslant 3$, $\mathrm{E}(X_1 X_j) = \dfrac{a(a-1)b}{n(n-1)(n-2)}$;

(b) $\displaystyle\sum_{i=2}^{n-2} \left(\sum_{j=i+2}^{n} \mathrm{E}(X_i X_j) \right) = \frac{a(a-1)b(b-1)}{2n(n-1)}$;

(c) $\mathrm{Var}\,(S) = \dfrac{a(a-1)b(b+1)}{n^2(n-1)}$.

13 **(i)** The continuous random variable X satisfies $0 \leqslant X \leqslant 1$, and has probability density function $\mathrm{f}(x)$ and cumulative distribution function $\mathrm{F}(x)$. The greatest value of $\mathrm{f}(x)$ is M, so that $0 \leqslant \mathrm{f}(x) \leqslant M$.

(a) Show that $0 \leqslant \mathrm{F}(x) \leqslant Mx$ for $0 \leqslant x \leqslant 1$.

(b) For any function $\mathrm{g}(x)$, show that

$$\int_0^1 2\mathrm{g}(x)\mathrm{F}(x)\mathrm{f}(x)\mathrm{d}x = \mathrm{g}(1) - \int_0^1 \mathrm{g}'(x)\big(\mathrm{F}(x)\big)^2\mathrm{d}x\,.$$

(i) The continuous random variable Y satisfies $0 \leqslant Y \leqslant 1$, and has probability density function $k\mathrm{F}(y)\mathrm{f}(y)$, where f and F are as above.

(a) Determine the value of the constant k.

(b) Show that

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leqslant \mathrm{E}(Y^n) \leqslant 2M\mu_{n+1}\,,$$

where $\mu_{n+1} = \mathrm{E}(X^{n+1})$ and $n \geqslant 0$.

(c) Hence show that, for $n \geqslant 1$,

$$\mu_n \geqslant \frac{n}{(n+1)M} - \frac{n-1}{n+1}\,.$$