Section A: Pure Mathematics

1 *All numbers referred to in this question are non-negative integers.*

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

2 **(i)** Show that $\int \ln(2-x)\,\mathrm{d}x = -(2-x)\ln(2-x) + (2-x) + c\,,$ where $x < 2$.

(ii) Sketch the curve A given by $y = \ln|x^2 - 4|$.

(iii) Show that the area of the finite region enclosed by the positive x-axis, the y-axis and the curve A is $4\ln(2+\sqrt{3}) - 2\sqrt{3}\,$.

(iv) The curve B is given by $y = \big|\ln|x^2 - 4|\big|\,$. Find the area between the curve B and the x-axis with $|x| < 2$.

[*Note: you may assume that* $t\ln t \to 0$ *as* $t \to 0$.]

3 The numbers a and b, where $b > a \geqslant 0$, are such that

$$\int_a^b x^2 \, \mathrm{d}x = \left(\int_a^b x \, \mathrm{d}x \right)^2 .$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0 \, .$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies this equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leqslant \frac{4}{3}$.

4 An accurate clock has an hour hand of length a and a minute hand of length b (where $b > a$), both measured from the pivot at the centre of the clock face. Let x be the distance between the ends of the hands when the angle between the hands is θ, where $0 \leqslant \theta < \pi$.

Show that the rate of increase of x is greatest when $x = (b^2 - a^2)^{\frac{1}{2}}$.

In the case when $b = 2a$ and the clock starts at mid-day (with both hands pointing vertically upwards), show that this occurs for the first time a little less than 11 minutes later.

5 **(i)** Let $\mathrm{f}(x) = (x + 2a)^3 - 27a^2 x$, where $a \geqslant 0$. By sketching $\mathrm{f}(x)$, show that $\mathrm{f}(x) \geqslant 0$ for $x \geqslant 0$.

(ii) Use part (i) to find the greatest value of xy^2 in the region of the x-y plane given by $x \geqslant 0$, $y \geqslant 0$ and $x + 2y \leqslant 3$. For what values of x and y is this greatest value achieved?

(iii) Use part (i) to show that $(p + q + r)^3 \geqslant 27pqr$ for any non-negative numbers p, q and r. If $(p + q + r)^3 = 27pqr$, what relationship must p, q and r satisfy?

6 **(i)** The sequence of numbers $u_0, u_1, \ldots$ is given by $u_0 = u$ and, for $n \geqslant 0$,

$$u_{n+1} = 4u_n(1 - u_n)\,. \qquad (*)$$

In the case $u = \sin^2\theta$ for some given angle θ, write down and simplify expressions for u_1 and u_2 in terms of θ. Conjecture an expression for u_n and prove your conjecture.

(ii) The sequence of numbers $v_0, v_1, \ldots$ is given by $v_0 = v$ and, for $n \geqslant 0$,

$$v_{n+1} = -pv_n^2 + qv_n + r\,,$$

where p, q and r are given numbers, with $p \neq 0$. Show that a substitution of the form $v_n = \alpha u_n + \beta$, where α and β are suitably chosen, results in the sequence $(*)$ provided that

$$4pr = 8 + 2q - q^2\,.$$

Hence obtain the sequence satisfying $v_0 = 1$ and, for $n \geqslant 0$, $v_{n+1} = -v_n^2 + 2v_n + 2\,$.

7 In the triangle OAB, the point D divides the side BO in the ratio $r : 1$ (so that $BD = rDO$), and the point E divides the side OA in the ratio $s : 1$ (so that $OE = sEA$), where r and s are both positive.

(i) The lines AD and BE intersect at G. Show that

$$\mathbf{g} = \frac{rs}{1 + r + rs}\,\mathbf{a} + \frac{1}{1 + r + rs}\,\mathbf{b}\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}$ are the position vectors with respect to O of A, B and G, respectively.

(ii) The line through G and O meets AB at F. Given that F divides AB in the ratio $t : 1$, find an expression for t in terms of r and s.

8 Let L_a denote the line joining the points $(a, 0)$ and $(0, 1 - a)$, where $0 < a < 1$. The line L_b is defined similarly.

(i) Determine the point of intersection of L_a and L_b, where $a \neq b$.

(ii) Show that this point of intersection, in the limit as $b \to a$, lies on the curve C given by

$$y = (1 - \sqrt{x})^2 \qquad (0 < x < 1)\,.$$

(iii) Show that every tangent to C is of the form L_a for some a.

Section B: Mechanics

9 A particle of mass m is projected due east at speed U from a point on horizontal ground at an angle θ above the horizontal, where $0 < \theta < 90^\circ$. In addition to the gravitational force mg, it experiences a horizontal force of magnitude mkg, where k is a positive constant, acting due west in the plane of motion of the particle. Determine expressions in terms of U, θ and g for the time, T_H, at which the particle reaches its greatest height and the time, T_L, at which it lands.

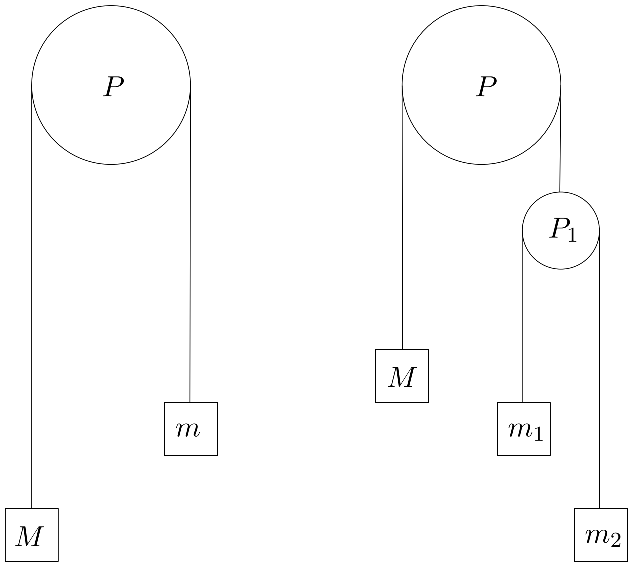
Let $T = U\cos\theta/(kg)$. By considering the relative magnitudes of T_H, T_L and T, or otherwise, sketch the trajectory of the particle in the cases $k\tan\theta < \frac{1}{2}$, $\frac{1}{2} < k\tan\theta < 1$, and $k\tan\theta > 1$. What happens when $k\tan\theta = 1$?

10 **(i)** A uniform spherical ball of mass M and radius R is released from rest with its centre a distance $H + R$ above horizontal ground. The coefficient of restitution between the ball and the ground is e. Show that, after bouncing, the centre of the ball reaches a height $R + He^2$ above the ground.

(ii) A second uniform spherical ball, of mass m and radius r, is now released from rest together with the first ball (whose centre is again a distance $H + R$ above the ground when it is released). The two balls are initially one on top of the other, with the second ball (of mass m) above the first. The two balls separate slightly during their fall, with their centres remaining in the same vertical line, so that they collide immediately after the first ball has bounced on the ground. The coefficient of restitution between the balls is also e. The centre of the second ball attains a height h above the ground.

Given that $R = 0.2$, $r = 0.05$, $H = 1.8$, $h = 4.5$ and $e = \frac{2}{3}$, determine the value of M/m.

11 The diagrams below show two separate systems of particles, strings and pulleys. In both systems, the pulleys are smooth and light, the strings are light and inextensible, the particles move vertically and the pulleys labelled with P are fixed. The masses of the particles are as indicated on the diagrams.



System I System II

(i) For system I show that the acceleration, a_1, of the particle of mass M, measured in the downwards direction, is given by

$$a_1 = \frac{M - m}{M + m}\, g \,,$$

where g is the acceleration due to gravity. Give an expression for the force on the pulley due to the tension in the string.

(ii) For system II show that the acceleration, a_2, of the particle of mass M, measured in the downwards direction, is given by

$$a_2 = \frac{M - 4\mu}{M + 4\mu}\, g \,,$$

where $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$.

In the case $m = m_1 + m_2$, show that $a_1 = a_2$ if and only if $m_1 = m_2$.

Section C: Probability and Statistics

12 A game in a casino is played with a fair coin and an unbiased cubical die whose faces are labelled $1, 1, 1, 2, 2$ and 3. In each round of the game, the die is rolled once and the coin is tossed once. The outcome of the round is a random variable X. The value, x, of X is determined as follows. If the result of the toss is heads then $x = |ks - 1|$, and if the result of the toss is tails then $x = |k - s|$, where s is the number on the die and k is a given number. Show that $\mathrm{E}(X^2) = k + 13(k-1)^2/6$.

Given that both $\mathrm{E}(X^2)$ and $\mathrm{E}(X)$ are positive integers, and that k is a single-digit positive integer, determine the value of k, and write down the probability distribution of X.

A gambler pays £1 to play the game, which consists of two rounds. The gambler is paid:

£w, where w is an integer, if the sum of the outcomes of the two rounds exceeds 25;
£1 if the sum of the outcomes equals 25;
nothing if the sum of the outcomes is less that 25.

Find, in terms of w, an expression for the amount the gambler expects to be paid in a game, and deduce the maximum possible value of w, given that the casino's owners choose w so that the game is in their favour.

13 A continuous random variable X has a *triangular* distribution, which means that it has a probability density function of the form

$$\mathrm{f}(x) = \begin{cases} \mathrm{g}(x) & \text{for } a < x \leqslant c \\ \mathrm{h}(x) & \text{for } c \leqslant x < b \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathrm{g}(x)$ is an increasing linear function with $\mathrm{g}(a) = 0$, $\mathrm{h}(x)$ is a decreasing linear function with $\mathrm{h}(b) = 0$, and $\mathrm{g}(c) = \mathrm{h}(c)$.

Show that $\mathrm{g}(x) = \dfrac{2(x-a)}{(b-a)(c-a)}$ and find a similar expression for $\mathrm{h}(x)$.

(i) Show that the mean of the distribution is $\frac{1}{3}(a+b+c)$.

(ii) Find the median of the distribution in the different cases that arise.