Section A: Pure Mathematics

1 In the triangle ABC, the base AB is of length 1 unit and the angles at A and B are α and β respectively, where $0 < \alpha \leqslant \beta$. The points P and Q lie on the sides AC and BC respectively, with $AP = PQ = QB = x$. The line PQ makes an angle of θ with the line through P parallel to AB.

(i) Show that $x\cos\theta = 1 - x\cos\alpha - x\cos\beta$, and obtain an expression for $x\sin\theta$ in terms of x, α and β. Hence show that

$$\big(1 + 2\cos(\alpha+\beta)\big)x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0\,. \qquad (*)$$

Show that $(*)$ is also satisfied if P and Q lie on AC produced and BC produced, respectively. [By definition, P lies on AC produced if P lies on the line through A and C and the points are in the order A, C, P.]

(ii) State the condition on α and β for $(*)$ to be linear in x. If this condition does not hold (but the condition $0 < \alpha \leqslant \beta$ still holds), show that $(*)$ has distinct real roots.

(iii) Find the possible values of x in the two cases (a) $\alpha = \beta = 45°$ and (b) $\alpha = 30°$, $\beta = 90°$, and illustrate each case with a sketch.

2 This question concerns the inequality

$$\int_0^\pi \big(\mathrm{f}(x)\big)^2\,\mathrm{d}x \leqslant \int_0^\pi \big(\mathrm{f}'(x)\big)^2\,\mathrm{d}x\,. \qquad (*)$$

(i) Show that $(*)$ is satisfied in the case $\mathrm{f}(x) = \sin nx$, where n is a positive integer.

Show by means of counterexamples that $(*)$ is not necessarily satisfied if either $\mathrm{f}(0) \neq 0$ or $\mathrm{f}(\pi) \neq 0$.

(ii) You may now assume that $(*)$ is satisfied for any (differentiable) function f for which $\mathrm{f}(0) = \mathrm{f}(\pi) = 0$.

By setting $\mathrm{f}(x) = ax^2 + bx + c$, where a, b and c are suitably chosen, show that $\pi^2 \leqslant 10$.

By setting $\mathrm{f}(x) = p\sin\frac{1}{2}x + q\cos\frac{1}{2}x + r$, where p, q and r are suitably chosen, obtain another inequality for π.

Which of these inequalities leads to a better estimate for π^2 ?

3 **(i)** Show, geometrically or otherwise, that the shortest distance between the origin and the line $y = mx + c$, where $c \geqslant 0$, is $c(m^2 + 1)^{-\frac{1}{2}}$.

(ii) The curve C lies in the x-y plane. Let the line L be tangent to C at a point P on C, and let a be the shortest distance between the origin and L. The curve C has the property that the distance a is the same for all points P on C.

Let P be the point on C with coordinates $(x, y(x))$. Given that the tangent to C at P is not vertical, show that

$$(y - xy')^2 = a^2\big(1 + (y')^2\big)\,. \qquad (*)$$

By first differentiating $(*)$ with respect to x, show that either $y = mx \pm a(1 + m^2)^{\frac{1}{2}}$ for some m or $x^2 + y^2 = a^2$.

(iii) Now suppose that C (as defined above) is a continuous curve for $-\infty < x < \infty$, consisting of the arc of a circle and two straight lines. Sketch an example of such a curve which has a non-vertical tangent at each point.

4 **(i)** By using the substitution $u = 1/x$, show that for $b > 0$

$$\int_{1/b}^{b} \frac{x \ln x}{(a^2 + x^2)(a^2x^2 + 1)}\,\mathrm{d}x = 0\,.$$

(ii) By using the substitution $u = 1/x$, show that for $b > 0$,

$$\int_{1/b}^{b} \frac{\arctan x}{x}\,\mathrm{d}x = \frac{\pi \ln b}{2}\,.$$

(iii) By using the result $\displaystyle\int_0^{\infty} \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{\pi}{2a}$ (where $a > 0$), and a substitution of the form $u = k/x$, for suitable k, show that

$$\int_0^{\infty} \frac{1}{(a^2 + x^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3} \qquad (a > 0).$$

5 Given that $y = xu$, where u is a function of x, write down an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(i) Use the substitution $y = xu$ to solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y + x}{y - 2x}$$

given that the solution curve passes through the point $(1, 1)$.

Give your answer in the form of a quadratic in x and y.

(ii) Using the substitutions $x = X + a$ and $y = Y + b$ for appropriate values of a and b, or otherwise, solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - 2y - 4}{2x + y - 3},$$

given that the solution curve passes through the point $(1, 1)$.

6 By simplifying $\sin(r + \frac{1}{2})x - \sin(r - \frac{1}{2})x$ or otherwise show that, for $\sin\frac{1}{2}x \neq 0$,

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin(n + \frac{1}{2})x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}.$$

The functions S_n, for $n = 1, 2, \ldots$, are defined by

$$\mathrm{S}_n(x) = \sum_{r=1}^{n} \frac{1}{r}\sin rx \qquad (0 \leqslant x \leqslant \pi).$$

(i) Find the stationary points of $\mathrm{S}_2(x)$ for $0 \leqslant x \leqslant \pi$, and sketch this function.

(ii) Show that if $\mathrm{S}_n(x)$ has a stationary point at $x = x_0$, where $0 < x_0 < \pi$, then

$$\sin nx_0 = (1 - \cos nx_0)\tan\tfrac{1}{2}x_0$$

and hence that $\mathrm{S}_n(x_0) \geqslant \mathrm{S}_{n-1}(x_0)$. Deduce that if $\mathrm{S}_{n-1}(x) > 0$ for all x in the interval $0 < x < \pi$, then $\mathrm{S}_n(x) > 0$ for all x in this interval.

(iii) Prove that $\mathrm{S}_n(x) \geqslant 0$ for $n \geqslant 1$ and $0 \leqslant x \leqslant \pi$.

7 **(i)** The function f is defined by $\mathrm{f}(x) = |x - a| + |x - b|$, where $a < b$. Sketch the graph of $\mathrm{f}(x)$, giving the gradient in each of the regions $x < a$, $a < x < b$ and $x > b$. Sketch on the same diagram the graph of $\mathrm{g}(x)$, where $\mathrm{g}(x) = |2x - a - b|$.

What shape is the quadrilateral with vertices $(a, 0)$, $(b, 0)$, $(b, \mathrm{f}(b))$ and $(a, \mathrm{f}(a))$?

(ii) Show graphically that the equation

$$|x - a| + |x - b| = |x - c| \,,$$

where $a < b$, has 0, 1 or 2 solutions, stating the relationship of c to a and b in each case.

(iii) For the equation

$$|x - a| + |x - b| = |x - c| + |x - d| \,,$$

where $a < b$, $c < d$ and $d - c < b - a$, determine the number of solutions in the various cases that arise, stating the relationship between a, b, c and d in each case.

8 For positive integers n, a and b, the integer c_r $(0 \leqslant r \leqslant n)$ is defined to be the coefficient of x^r in the expansion in powers of x of $(a + bx)^n$. Write down an expression for c_r in terms of r, n, a and b.

For given n, a and b, let m denote a value of r for which c_r is greatest (that is, $c_m \geqslant c_r$ for $0 \leqslant r \leqslant n$).

Show that

$$\frac{b(n+1)}{a+b} - 1 \leqslant m \leqslant \frac{b(n+1)}{a+b} \,.$$

Deduce that m is either a unique integer or one of two consecutive integers.

Let $\mathrm{G}(n, a, b)$ denote the unique value of m (if there is one) or the larger of the two possible values of m.

(i) Evaluate $\mathrm{G}(9, 1, 3)$ and $\mathrm{G}(9, 2, 3)$.

(ii) For any positive integer k, find $\mathrm{G}(2k, a, a)$ and $\mathrm{G}(2k - 1, a, a)$ in terms of k.

(iii) For fixed n and b, determine a value of a for which $\mathrm{G}(n, a, b)$ is greatest.

(iv) For fixed n, find the greatest possible value of $\mathrm{G}(n, 1, b)$. For which values of b is this greatest value achieved?

Section B: Mechanics

9 A uniform rectangular lamina $ABCD$ rests in equilibrium in a vertical plane with the corner A in contact with a rough vertical wall. The plane of the lamina is perpendicular to the wall. It is supported by a light inextensible string attached to the side AB at a distance d from A. The other end of the string is attached to a point on the wall above A where it makes an acute angle θ with the downwards vertical. The side AB makes an acute angle ϕ with the upwards vertical at A. The sides BC and AB have lengths $2a$ and $2b$ respectively. The coefficient of friction between the lamina and the wall is μ.

(i) Show that, when the lamina is in limiting equilibrium with the frictional force acting upwards,

$$d\sin(\theta+\phi) = (\cos\theta + \mu\sin\theta)(a\cos\phi + b\sin\phi)\,. \qquad (*)$$

(ii) How should $(*)$ be modified if the lamina is in limiting equilibrium with the frictional force acting downwards?

(iii) Find a condition on d, in terms of a, b, $\tan\theta$ and $\tan\phi$, which is necessary and sufficient for the frictional force to act upwards. Show that this condition cannot be satisfied if $b(2\tan\theta + \tan\phi) < a$.

10 A particle is projected from a point O on horizontal ground with initial speed u and at an angle of θ above the ground. The motion takes place in the x-y plane, where the x-axis is horizontal, the y-axis is vertical and the origin is O. Obtain the Cartesian equation of the particle's trajectory in terms of u, g and λ, where $\lambda = \tan\theta$.

Now consider the trajectories for different values of θ with u fixed. Show that for a given value of x, the coordinate y can take all values up to a maximum value, Y, which you should determine as a function of x, u and g.

Sketch a graph of Y against x and indicate on your graph the set of points that can be reached by a particle projected from O with speed u.

Hence find the furthest distance from O that can be achieved by such a projectile.

11 A small smooth ring R of mass m is free to slide on a fixed smooth horizontal rail. A light inextensible string of length L is attached to one end, O, of the rail. The string passes through the ring, and a particle P of mass km (where $k > 0$) is attached to its other end; this part of the string hangs at an acute angle α to the vertical and it is given that α is constant in the motion.

Let x be the distance between O and the ring. Taking the y-axis to be vertically upwards, write down the Cartesian coordinates of P relative to O in terms of x, L and α.

(i) By considering the vertical component of the equation of motion of P, show that

$$km\ddot{x}\cos\alpha = T\cos\alpha - kmg\,,$$

where T is the tension in the string. Obtain two similar equations relating to the horizontal components of the equations of motion of P and R.

(ii) Show that $\dfrac{\sin\alpha}{(1-\sin\alpha)^2} = k$, and deduce, by means of a sketch or otherwise, that motion with α constant is possible for all values of k.

(iii) Show that $\ddot{x} = -g\tan\alpha\,.$

Section C: Probability and Statistics

12 The lifetime of a fly (measured in hours) is given by the continuous random variable T with probability density function $\mathrm{f}(t)$ and cumulative distribution function $\mathrm{F}(t)$. The *hazard function*, $\mathrm{h}(t)$, is defined, for $\mathrm{F}(t) < 1$, by

$$\mathrm{h}(t) = \frac{\mathrm{f}(t)}{1 - \mathrm{F}(t)}\,.$$

(i) Given that the fly lives to at least time t, show that the probability of its dying within the following δt is approximately $\mathrm{h}(t)\,\delta t$ for small values of δt.

(ii) Find the hazard function in the case $\mathrm{F}(t) = t/a$ for $0 < t < a$. Sketch $\mathrm{f}(t)$ and $\mathrm{h}(t)$ in this case.

(iii) The random variable T is distributed on the interval $t > a$, where $a > 0$, and its hazard function is t^{-1}. Determine the probability density function for T.

(iv) Show that $\mathrm{h}(t)$ is constant for $t > b$ and zero otherwise if and only if $\mathrm{f}(t) = k\mathrm{e}^{-k(t-b)}$ for $t > b$, where k is a positive constant.

(v) The random variable T is distributed on the interval $t > 0$ and its hazard function is given by

$$\mathrm{h}(t) = \left(\frac{\lambda}{\theta^{\lambda}}\right) t^{\lambda - 1}\,,$$

where λ and θ are positive constants. Find the probability density function for T.

13 A random number generator prints out a sequence of integers I_1, I_2, I_3, ... Each integer is independently equally likely to be any one of 1, 2, ..., n, where n is fixed. The random variable X takes the value r, where I_r is the first integer which is a repeat of some earlier integer.

Write down an expression for $\mathrm{P}(X = 4)$.

(i) Find an expression for $\mathrm{P}(X = r)$, where $2 \leqslant r \leqslant n+1$. Hence show that, for any positive integer n,

$$\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{2}{n} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots \;=\; 1\,.$$

(ii) Write down an expression for $\mathrm{E}(X)$. (You do not need to simplify it.)

(iii) Write down an expression for $\mathrm{P}(X \geqslant k)$.

(iv) Show that, for any discrete random variable Y taking the values 1, 2, ..., N,

$$\mathrm{E}(Y) = \sum_{k=1}^{N} \mathrm{P}(Y \geqslant k)\,.$$

Hence show that, for any positive integer n,

$$\left(1 - \frac{1^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3^2}{n}\right) + \cdots \;=\; 0.$$