Section A: Pure Mathematics

1 Let a, b and c be real numbers such that $a + b + c = 0$ and let

$$(1 + ax)(1 + bx)(1 + cx) = 1 + qx^2 + rx^3$$

for all real x. Show that $q = bc + ca + ab$ and $r = abc$.

(i) Show that the coefficient of x^n in the series expansion (in ascending powers of x) of $\ln(1 + qx^2 + rx^3)$ is $(-1)^{n+1}S_n$ where

$$S_n = \frac{a^n + b^n + c^n}{n}\,, \qquad (n \geqslant 1).$$

(ii) Find, in terms of q and r, the coefficients of x^2, x^3 and x^5 in the series expansion (in ascending powers of x) of $\ln(1 + qx^2 + rx^3)$ and hence show that $S_2S_3 = S_5$.

(iii) Show that $S_2S_5 = S_7$.

(iv) Give a proof of, or find a counterexample to, the claim that $S_2S_7 = S_9$.

2 **(i)** Show, by means of the substitution $u = \cosh x$, that

$$\int \frac{\sinh x}{\cosh 2x}\,\mathrm{d}x = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{2}\cosh x - 1}{\sqrt{2}\cosh x + 1} \right| + C\,.$$

(ii) Use a similar substitution to find an expression for

$$\int \frac{\cosh x}{\cosh 2x}\,\mathrm{d}x\,.$$

(iii) Using parts (i) and (ii) above, show that

$$\int_0^1 \frac{1}{1+u^4}\,\mathrm{d}u = \frac{\pi + 2\ln(\sqrt{2}+1)}{4\sqrt{2}}\,.$$

3 **(i)** The line L has equation $y = mx + c$, where $m > 0$ and $c > 0$. Show that, in the case $mc > a > 0$, the shortest distance between L and the parabola $y^2 = 4ax$ is

$$\frac{mc - a}{m\sqrt{m^2 + 1}}\,.$$

What is the shortest distance in the case that $mc \leqslant a$?

(ii) Find the shortest distance between the point $(p, 0)$, where $p > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the value of p/a. [*You may wish to use the parametric coordinates* $(at^2, 2at)$ *of points on the parabola.*]

Hence find the shortest distance between the circle $(x - p)^2 + y^2 = b^2$, where $p > 0$ and $b > 0$, and the parabola $y^2 = 4ax$, where $a > 0$, in the different cases that arise according to the values of p, a and b.

4 **(i)** Let

$$I = \int_0^1 \left((y')^2 - y^2\right)\,\mathrm{d}x \qquad \text{and} \qquad I_1 = \int_0^1 (y' + y\tan x)^2\,\mathrm{d}x\,,$$

where y is a given function of x satisfying $y = 0$ at $x = 1$. Show that $I - I_1 = 0$ and deduce that $I \geqslant 0$. Show further that $I = 0$ only if $y = 0$ for all x ($0 \leqslant x \leqslant 1$).

(ii) Let

$$J = \int_0^1 \left((y')^2 - a^2y^2\right)\,\mathrm{d}x\,,$$

where a is a given positive constant and y is a given function of x, not identically zero, satisfying $y = 0$ at $x = 1$. By considering an integral of the form

$$\int_0^1 (y' + ay\tan bx)^2\,\mathrm{d}x\,,$$

where b is suitably chosen, show that $J \geqslant 0$. You should state the range of values of a, in the form $a < k$, for which your proof is valid.

In the case $a = k$, find a function y (not everywhere zero) such that $J = 0$.

5 A quadrilateral drawn in the complex plane has vertices A, B, C and D, labelled anticlockwise. These vertices are represented, respectively, by the complex numbers a, b, c and d. Show that $ABCD$ is a parallelogram (defined as a quadrilateral in which opposite sides are parallel and equal in length) if and only if $a+c=b+d$. Show further that, in this case, $ABCD$ is a square if and only if $\mathrm{i}(a-c)=b-d$.

Let $PQRS$ be a quadrilateral in the complex plane, with vertices labelled anticlockwise, the internal angles of which are all less than $180°$. Squares with centres X, Y, Z and T are constructed externally to the quadrilateral on the sides PQ, QR, RS and SP, respectively.

(i) If P and Q are represented by the complex numbers p and q, respectively, show that X can be represented by

$$\tfrac{1}{2}\big(p(1+\mathrm{i})+q(1-\mathrm{i})\big)\,.$$

(ii) Show that $XYZT$ is a square if and only if $PQRS$ is a parallelogram.

6 Starting from the result that

$$\mathrm{h}(t)>0 \text{ for } 0<t<x \Longrightarrow \int_0^x \mathrm{h}(t)\,\mathrm{d}t>0\,,$$

show that, if $\mathrm{f}''(t)>0$ for $0<t<x_0$ and $\mathrm{f}(0)=\mathrm{f}'(0)=0$, then $\mathrm{f}(t)>0$ for $0<t<x_0$.

(i) Show that, for $0<x<\tfrac{1}{2}\pi$,

$$\cos x\cosh x<1\,.$$

(ii) Show that, for $0<x<\tfrac{1}{2}\pi$,

$$\frac{1}{\cosh x}<\frac{\sin x}{x}<\frac{x}{\sinh x}\,.$$

7 The four distinct points P_i ($i = 1, 2, 3, 4$) are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines P_1P_3 and P_2P_4 intersect at Q.

(i) By considering the triangles P_1QP_4 and P_2QP_3 show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)$.

(ii) Let $\mathbf{p}_i$ be the position vector of the point P_i ($i = 1, 2, 3, 4$). Show that there exist numbers a_i, not all zero, such that

$$\sum_{i=1}^{4} a_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} a_i \mathbf{p}_i = \mathbf{0}\,. \qquad (*)$$

(iii) Let a_i ($i = 1, 2, 3, 4$) be any numbers, not all zero, that satisfy $(*)$. Show that $a_1 + a_3 \neq 0$ and that the lines P_1P_3 and P_2P_4 intersect at the point with position vector

$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}\,.$$

Deduce that $a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2$.

8 The numbers $\mathrm{f}(r)$ satisfy $\mathrm{f}(r) > \mathrm{f}(r+1)$ for $r = 1, 2, \ldots$. Show that, for any non-negative integer n,

$$k^n(k-1)\,\mathrm{f}(k^{n+1}) \leqslant \sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) \leqslant k^n(k-1)\,\mathrm{f}(k^n)$$

where k is an integer greater than 1.

(i) By taking $\mathrm{f}(r) = 1/r$, show that

$$\frac{N+1}{2} \leqslant \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leqslant N+1\,.$$

Deduce that the sum $\sum\limits_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $\mathrm{f}(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leqslant 1\tfrac{1}{3}\,.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n.

Section B: Mechanics

9 A particle of mass m is projected with velocity $\mathbf{u}$. It is acted upon by the force $m\mathbf{g}$ due to gravity and by a resistive force $-mk\mathbf{v}$, where $\mathbf{v}$ is its velocity and k is a positive constant.

Given that, at time t after projection, its position $\mathbf{r}$ relative to the point of projection is given by

$$\mathbf{r} = \frac{kt - 1 + \mathrm{e}^{-kt}}{k^2}\,\mathbf{g} + \frac{1 - \mathrm{e}^{-kt}}{k}\,\mathbf{u}\,,$$

find an expression for $\mathbf{v}$ in terms of k, t, $\mathbf{g}$ and $\mathbf{u}$. Verify that the equation of motion and the initial conditions are satisfied.

Let $\mathbf{u} = u\cos\alpha\,\mathbf{i} + u\sin\alpha\,\mathbf{j}$ and $\mathbf{g} = -g\,\mathbf{j}$, where $0 < \alpha < 90°$, and let T be the time after projection at which $\mathbf{r}\,.\,\mathbf{j} = 0$. Show that

$$uk\sin\alpha = \left(\frac{kT}{1 - \mathrm{e}^{-kT}} - 1\right) g\,.$$

Let β be the acute angle between $\mathbf{v}$ and $\mathbf{i}$ at time T. Show that

$$\tan\beta = \frac{(\mathrm{e}^{kT} - 1)g}{uk\cos\alpha} - \tan\alpha\,.$$

Show further that $\tan\beta > \tan\alpha$ (you may assume that $\sinh kT > kT$) and deduce that $\beta > \alpha$.

10 Two particles X and Y, of equal mass m, lie on a smooth horizontal table and are connected by a light elastic spring of natural length a and modulus of elasticity λ. Two more springs, identical to the first, connect X to a point P on the table and Y to a point Q on the table. The distance between P and Q is $3a$.

Initially, the particles are held so that $XP = a$, $YQ = \frac{1}{2}a$, and $PXYQ$ is a straight line. The particles are then released.

At time t, the particle X is a distance $a + x$ from P and the particle Y is a distance $a + y$ from Q. Show that

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\lambda}{a}(2x + y)$$

and find a similar expression involving $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}$. Deduce that

$$x - y = A\cos\omega t + B\sin\omega t$$

where A and B are constants to be determined and $ma\omega^2 = \lambda$. Find a similar expression for $x + y$.

Show that Y will never return to its initial position.

11 A particle P of mass m is connected by two light inextensible strings to two fixed points A and B, with A vertically above B. The string AP has length x. The particle is rotating about the vertical through A and B with angular velocity ω, and both strings are taut. Angles PAB and PBA are α and β, respectively.

Find the tensions T_A and T_B in the strings AP and BP (respectively), and hence show that $\omega^2 x \cos\alpha \geqslant g$.

Consider now the case that $\omega^2 x \cos\alpha = g$. Given that $AB = h$ and $BP = d$, where $h > d$, show that $h\cos\alpha \geqslant \sqrt{h^2 - d^2}$. Show further that

$$mg < T_A \leqslant \frac{mgh}{\sqrt{h^2 - d^2}}\,.$$

Describe the geometry of the strings when T_A attains its upper bound.

Section C: Probability and Statistics

12 The random variable X has probability density function $\mathrm{f}(x)$ (which you may assume is differentiable) and cumulative distribution function $\mathrm{F}(x)$ where $-\infty < x < \infty$. The random variable Y is defined by $Y = \mathrm{e}^X$. You may assume throughout this question that X and Y have unique modes.

(i) Find the median value y_m of Y in terms of the median value x_m of X.

(ii) Show that the probability density function of Y is $\mathrm{f}(\ln y)/y$, and deduce that the mode λ of Y satisfies $\mathrm{f}'(\ln\lambda) = \mathrm{f}(\ln\lambda)$.

(iii) Suppose now that $X \sim \mathrm{N}(\mu, \sigma^2)$, so that

$$\mathrm{f}(x) = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)} .$$

Explain why

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-(x-\mu-\sigma^2)^2/(2\sigma^2)}\,\mathrm{d}x = 1$$

and hence show that $\mathrm{E}(Y) = \mathrm{e}^{\mu+\frac{1}{2}\sigma^2}$.

(iv) Show that, when $X \sim \mathrm{N}(\mu, \sigma^2)$,

$$\lambda < y_m < \mathrm{E}(Y) .$$

13 I play a game which has repeated rounds. Before the first round, my score is 0. Each round can have three outcomes:

1. my score is unchanged and the game ends;
2. my score is unchanged and I continue to the next round;
3. my score is increased by one and I continue to the next round.

The probabilities of these outcomes are a, b and c, respectively (the same in each round), where $a + b + c = 1$ and $abc \neq 0$. The random variable N represents my score at the end of a randomly chosen game.

Let $\mathrm{G}(t)$ be the probability generating function of N.

(i) Suppose in the first round, the game ends. Show that the probability generating function conditional on this happening is 1.

(ii) Suppose in the first round, the game continues to the next round with no change in score. Show that the probability generating function conditional on this happening is $\mathrm{G}(t)$.

(iii) By comparing the coefficients of t^n, show that $\mathrm{G}(t) = a + b\mathrm{G}(t) + ct\mathrm{G}(t)$. Deduce that, for $n \geqslant 0$,

$$P(N = n) = \frac{ac^n}{(1-b)^{n+1}} \,.$$

(iv) Show further that, for $n \geqslant 0$,

$$P(N = n) = \frac{\mu^n}{(1+\mu)^{n+1}} \,,$$

where $\mu = \mathrm{E}(N)$.