Section A: Pure Mathematics

1 **(i)** Let

$$I_n = \int_0^\infty \frac{1}{(1+u^2)^n}\,\mathrm{d}u\,,$$

where n is a positive integer. Show that

$$I_n - I_{n+1} = \frac{1}{2n} I_n$$

and deduce that

$$I_{n+1} = \frac{(2n)!\,\pi}{2^{2n+1}(n!)^2}\,.$$

(ii) Let

$$J = \int_0^\infty \mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x\,,$$

where f is any function for which the integral exists. Show that

$$J = \int_0^\infty x^{-2}\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x \;= \frac{1}{2}\int_0^\infty (1+x^{-2})\mathrm{f}\big((x-x^{-1})^2\big)\,\mathrm{d}x \;= \int_0^\infty \mathrm{f}\big(u^2\big)\,\mathrm{d}u\,.$$

(iii) Hence evaluate

$$\int_0^\infty \frac{x^{2n-2}}{(x^4-x^2+1)^n}\,\mathrm{d}x\,,$$

where n is a positive integer.

2 If s_1, s_2, s_3, ... and t_1, t_2, t_3, ... are sequences of positive numbers, we write

$$(s_n) \leqslant (t_n)$$

to mean

"there exists a positive integer m such that $s_n \leqslant t_n$ whenever $n \geqslant m$".

Determine whether each of the following statements is true or false. In the case of a true statement, you should give a proof which includes an explicit determination of an appropriate m; in the case of a false statement, you should give a counterexample.

(i) $(1000n) \leqslant (n^2)$.

(ii) If it is not the case that $(s_n) \leqslant (t_n)$, then it is the case that $(t_n) \leqslant (s_n)$.

(iii) If $(s_n) \leqslant (t_n)$ and $(t_n) \leqslant (u_n)$, then $(s_n) \leqslant (u_n)$.

(iv) $(n^2) \leqslant (2^n)$.

3 In this question, r and θ are polar coordinates with $r \geqslant 0$ and $-\pi < \theta \leqslant \pi$, and a and b are positive constants.

Let L be a fixed line and let A be a fixed point not lying on L. Then the locus of points that are a fixed distance (call it d) from L measured along lines through A is called a *conchoid of Nicomedes*.

(i) Show that if

$$|r - a\sec\theta| = b\,, \qquad (*)$$

where $a > b$, then $\sec\theta > 0$. Show that all points with coordinates satisfying $(*)$ lie on a certain conchoid of Nicomedes (you should identify L, d and A). Sketch the locus of these points.

(ii) In the case $a < b$, sketch the curve (including the loop for which $\sec\theta < 0$) given by

$$|r - a\sec\theta| = b\,.$$

Find the area of the loop in the case $a = 1$ and $b = 2$.

[Note: $\int \sec\theta \,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.]

4 **(i)** If a, b and c are all real, show that the equation

$$z^3 + az^2 + bz + c = 0 \qquad (*)$$

has at least one real root.

(ii) Let

$$S_1 = z_1 + z_2 + z_3, \quad S_2 = z_1^2 + z_2^2 + z_3^2, \quad S_3 = z_1^3 + z_2^3 + z_3^3\,,$$

where z_1, z_2 and z_3 are the roots of the equation $(*)$. Express a and b in terms of S_1 and S_2, and show that

$$6c = -S_1^3 + 3S_1S_2 - 2S_3\,.$$

(iii) The six real numbers r_k and θ_k ($k = 1,\ 2,\ 3$), where $r_k > 0$ and $-\pi < \theta_k < \pi$, satisfy

$$\sum_{k=1}^{3} r_k \sin(\theta_k) = 0\,, \quad \sum_{k=1}^{3} r_k^2 \sin(2\theta_k) = 0\,, \quad \sum_{k=1}^{3} r_k^3 \sin(3\theta_k) = 0\,.$$

Show that $\theta_k = 0$ for at least one value of k.

Show further that if $\theta_1 = 0$ then $\theta_2 = -\theta_3$.

5 **(i)** In the following argument to show that $\sqrt{2}$ is irrational, give proofs appropriate for steps 3, 5 and 6.

1. Assume that $\sqrt{2}$ is rational.
2. Define the set S to be the set of positive integers with the following property:

   n is in S if and only if $n\sqrt{2}$ is an integer.

3. Show that the set S contains at least one positive integer.
4. Define the integer k to be the smallest positive integer in S.
5. Show that $(\sqrt{2} - 1)k$ is in S.
6. Show that steps 4 and 5 are contradictory and hence that $\sqrt{2}$ is irrational.

(ii) Prove that $2^{\frac{1}{3}}$ is rational if and only if $2^{\frac{2}{3}}$ is rational.

Use an argument similar to that of part (i) to prove that $2^{\frac{1}{3}}$ and $2^{\frac{2}{3}}$ are irrational.

6 **(i)** Let w and z be complex numbers, and let $u = w + z$ and $v = w^2 + z^2$. Prove that w and z are real if and only if u and v are real and $u^2 \leqslant 2v$.

(ii) The complex numbers u, w and z satisfy the equations

$$\begin{aligned} w + z - u &= 0 \\ w^2 + z^2 - u^2 &= -\tfrac{2}{3} \\ w^3 + z^3 - \lambda u &= -\lambda \end{aligned}$$

where λ is a positive real number. Show that for all values of λ except one (which you should find) there are three possible values of u, all real.

Are w and z necessarily real? Give a proof or counterexample.

7 An operator D is defined, for any function f, by

$$\mathrm{Df}(x) = x\frac{\mathrm{df}(x)}{\mathrm{d}x}.$$

The notation D^n means that D is applied n times; for example

$$\mathrm{D}^2\mathrm{f}(x) = x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{df}(x)}{\mathrm{d}x}\right).$$

Show that, for any constant a, $\mathrm{D}^2 x^a = a^2 x^a$.

(i) Show that if $\mathrm{P}(x)$ is a polynomial of degree r (where $r \geqslant 1$) then, for any positive integer n, $\mathrm{D}^n\mathrm{P}(x)$ is also a polynomial of degree r.

(ii) Show that if n and m are positive integers with $n < m$, then $\mathrm{D}^n(1-x)^m$ is divisible by $(1-x)^{m-n}$.

(iii) Deduce that, if m and n are positive integers with $n < m$, then

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}r^n = 0\,.$$

8 **(i)** Show that under the changes of variable $x = r\cos\theta$ and $y = r\sin\theta$, where r is a function of θ with $r > 0$, the differential equation

$$(y+x)\frac{\mathrm{d}y}{\mathrm{d}x} = y - x$$

becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} + r = 0\,.$$

Sketch a solution in the x-y plane.

(ii) Show that the solutions of

$$\big(y + x - x(x^2+y^2)\big)\ \frac{\mathrm{d}y}{\mathrm{d}x} = y - x - y(x^2+y^2)$$

can be written in the form

$$r^2 = \frac{1}{1 + A\mathrm{e}^{2\theta}}$$

and sketch the different forms of solution that arise according to the value of A.

Section B: Mechanics

9 A particle P of mass m moves on a smooth fixed straight horizontal rail and is attached to a fixed peg Q by a light elastic string of natural length a and modulus λ. The peg Q is a distance a from the rail. Initially P is at rest with $PQ = a$.

An impulse imparts to P a speed v along the rail. Let x be the displacement at time t of P from its initial position. Obtain the equation

$$\dot{x}^2 = v^2 - k^2 \left(\sqrt{x^2 + a^2} - a \right)^2$$

where $k^2 = \lambda/(ma)$, $k > 0$ and the dot denotes differentiation with respect to t.

Find, in terms of k, a and v, the greatest value, x_0, attained by x. Find also the acceleration of P at $x = x_0$.

Obtain, in the form of an integral, an expression for the period of the motion. Show that in the case $v \ll ka$ (that is, v is much less than ka), this is approximately

$$\sqrt{\frac{32a}{kv}} \int_0^1 \frac{1}{\sqrt{1 - u^4}} \, \mathrm{d}u \, .$$

10 A light rod of length $2a$ has a particle of mass m attached to each end and it moves in a vertical plane. The midpoint of the rod has coordinates (x, y), where the x-axis is horizontal (within the plane of motion) and y is the height above a horizontal table. Initially, the rod is vertical, and at time t later it is inclined at an angle θ to the vertical.

Show that the velocity of one particle can be written in the form

$$\begin{pmatrix} \dot{x} + a\dot{\theta}\cos\theta \\ \dot{y} - a\dot{\theta}\sin\theta \end{pmatrix}$$

and that

$$m \begin{pmatrix} \ddot{x} + a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta \\ \ddot{y} - a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta \end{pmatrix} = -T \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

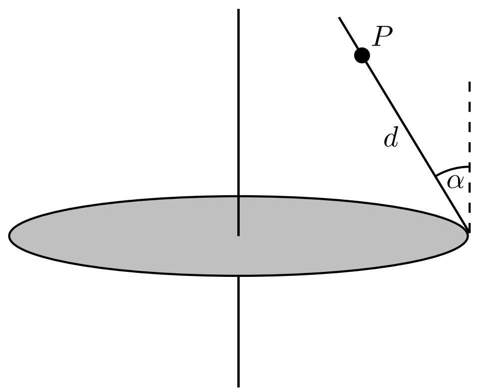
where the dots denote differentiation with respect to time t and T is the tension in the rod. Obtain the corresponding equations for the other particle.

Deduce that $\ddot{x} = 0$, $\ddot{y} = -g$ and $\ddot{\theta} = 0$.

Initially, the midpoint of the rod is a height h above the table, the velocity of the higher particle is $\begin{pmatrix} u \\ v \end{pmatrix}$, and the velocity of the lower particle is $\begin{pmatrix} 0 \\ v \end{pmatrix}$. Given that the two particles hit the table for the first time simultaneously, when the rod has rotated by $\frac{1}{2}\pi$, show that

$$2hu^2 = \pi^2 a^2 g - 2\pi uva \, .$$

11 **(i)** A horizontal disc of radius r rotates about a vertical axis through its centre with angular speed ω. One end of a light rod is fixed by a smooth hinge to the edge of the disc so that it can rotate freely in a vertical plane through the centre of the disc. A particle P of mass m is attached to the rod at a distance d from the hinge. The rod makes a constant angle α with the upward vertical, as shown in the diagram, and $d \sin\alpha < r$.



By considering moments about the hinge for the (light) rod, show that the force exerted on the rod by P is parallel to the rod.

Show also that

$$r \cot\alpha = a + d\cos\alpha\,,$$

where $a = \dfrac{g}{\omega^2}$. State clearly the direction of the force exerted by the hinge on the rod, and find an expression for its magnitude in terms of m, g and α.

(ii) The disc and rod rotate as in part (i), but two particles (instead of P) are attached to the rod. The masses of the particles are m_1 and m_2 and they are attached to the rod at distances d_1 and d_2 from the hinge, respectively. The rod makes a constant angle β with the upward vertical and $d_1 \sin\beta < d_2 \sin\beta < r$. Show that β satisfies an equation of the form

$$r \cot\beta = a + b\cos\beta\,,$$

where b should be expressed in terms of d_1, d_2, m_1 and m_2.

Section C: Probability and Statistics

12 A 6-sided fair die has the numbers 1, 2, 3, 4, 5, 6 on its faces. The die is thrown n times, the outcome (the number on the top face) of each throw being independent of the outcome of any other throw. The random variable S_n is the sum of the outcomes.

(i) The random variable R_n is the remainder when S_n is divided by 6. Write down the probability generating function, $\mathrm{G}(x)$, of R_1 and show that the probability generating function of R_2 is also $\mathrm{G}(x)$. Use a generating function to find the probability that S_n is divisible by 6.

(ii) The random variable T_n is the remainder when S_n is divided by 5. Write down the probability generating function, $\mathrm{G}_1(x)$, of T_1 and show that $\mathrm{G}_2(x)$, the probability generating function of T_2, is given by

$$\mathrm{G}_2(x) = \tfrac{1}{36}(x^2 + 7y)$$

where $y = 1 + x + x^2 + x^3 + x^4$.

Obtain the probability generating function of T_n and hence show that the probability that S_n is divisible by 5 is

$$\frac{1}{5}\left(1 - \frac{1}{6^n}\right)$$

if n is not divisible by 5. What is the corresponding probability if n is divisible by 5?

13 Each of the two independent random variables X and Y is uniformly distributed on the interval $[0, 1]$.

(i) By considering the lines $x+y = \text{constant}$ in the x-y plane, find the cumulative distribution function of $X + Y$.

(ii) Hence show that the probability density function f of $(X + Y)^{-1}$ is given by

$$\mathrm{f}(t) = \begin{cases} 2t^{-2} - t^{-3} & \text{for } \frac{1}{2} \leqslant t \leqslant 1 \\ t^{-3} & \text{for } 1 \leqslant t < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Evaluate $\mathrm{E}\left(\dfrac{1}{X+Y}\right)$.

(iii) Find the cumulative distribution function of Y/X and use this result to find the probability density function of $\dfrac{X}{X+Y}$.

Write down $\mathrm{E}\left(\dfrac{X}{X+Y}\right)$ and verify your result by integration.