Section A: Pure Mathematics

1 **(i)** For $n = 1$, 2, 3 and 4, the functions p_n and q_n are defined by

$$\mathrm{p}_n(x) = (x+1)^{2n} - (2n+1)x(x^2+x+1)^{n-1}$$

and

$$\mathrm{q}_n(x) = \frac{x^{2n+1}+1}{x+1} \qquad (x \neq -1)\,.$$

Show that $\mathrm{p}_n(x) \equiv \mathrm{q}_n(x)$ (for $x \neq -1$) in the cases $n = 1$, $n = 2$ and $n = 3$.

Show also that this does not hold in the case $n = 4$.

(ii) Using results from part (i):

(a) express $\dfrac{300^3+1}{301}$ as the product of two factors (neither of which is 1);

(b) express $\dfrac{7^{49}+1}{7^7+1}$ as the product of two factors (neither of which is 1), each written in terms of various powers of 7 which you should not attempt to calculate explicitly.

2 Differentiate, with respect to x,

$$(ax^2+bx+c)\,\ln\big(x+\sqrt{1+x^2}\big) + \big(dx+e\big)\sqrt{1+x^2}\,,$$

where a, b, c, d and e are constants. You should simplify your answer as far as possible.

Hence integrate:

(i) $\ln\big(x+\sqrt{1+x^2}\,\big)$;

(ii) $\sqrt{1+x^2}$;

(iii) $x\ln\big(x+\sqrt{1+x^2}\,\big)$.

3 In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that (for example) $\lfloor 2.9 \rfloor = 2$, $\lfloor 2 \rfloor = 2$ and $\lfloor -1.5 \rfloor = -2$.

On separate diagrams draw the graphs, for $-\pi \leqslant x \leqslant \pi$, of:

(i) $y = \lfloor x \rfloor$; (ii) $y = \sin\lfloor x \rfloor$; (iii) $y = \lfloor \sin x \rfloor$; (iv) $y = \lfloor 2\sin x \rfloor$.

In each case, you should indicate clearly the value of y at points where the graph is discontinuous.

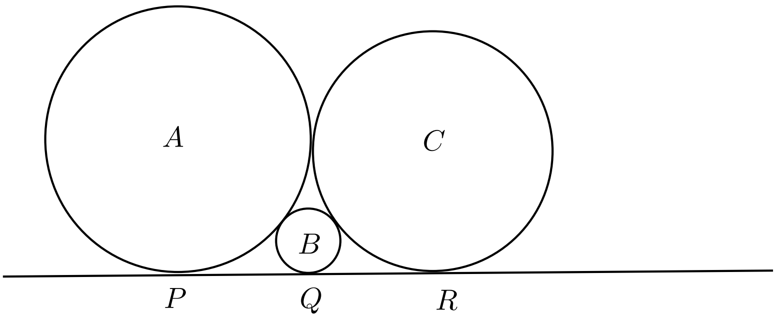
4 **(i)** Differentiate $\dfrac{z}{(1+z^2)^{\frac{1}{2}}}$ with respect to z.

(ii) The *signed curvature* κ of the curve $y = \mathrm{f}(x)$ is defined by

$$\kappa = \frac{\mathrm{f}''(x)}{\big(1 + (\mathrm{f}'(x))^2\big)^{\frac{3}{2}}}.$$

Use this definition to determine all curves for which the signed curvature is a non-zero constant. For these curves, what is the geometrical significance of κ?

5 **(i)**



The diagram shows three touching circles A, B and C, with a common tangent PQR. The radii of the circles are a, b and c, respectively.

Show that

$$\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}} \qquad (*)$$

and deduce that

$$2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \qquad (**)$$

(ii) Instead, let a, b and c be positive numbers, with $b < c < a$, which satisfy $(**)$. Show that they also satisfy $(*)$.

6 The sides OA and CB of the quadrilateral $OABC$ are parallel. The point X lies on OA, between O and A. The position vectors of A, B, C and X relative to the origin O are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{x}$, respectively. Explain why $\mathbf{c}$ and $\mathbf{x}$ can be written in the form

$$\mathbf{c} = k\mathbf{a} + \mathbf{b} \qquad \text{and} \qquad \mathbf{x} = m\mathbf{a}\,,$$

where k and m are scalars, and state the range of values that each of k and m can take.

The lines OB and AC intersect at D, the lines XD and BC intersect at Y and the lines OY and AB intersect at Z. Show that the position vector of Z relative to O can be written as

$$\frac{\mathbf{b} + mk\mathbf{a}}{mk+1}\,.$$

The lines DZ and OA intersect at T. Show that

$$OT \times OA = OX \times TA \qquad \text{and} \qquad \frac{1}{OT} = \frac{1}{OX} + \frac{1}{OA}\,,$$

where, for example, OT denotes the length of the line joining O and T.

7 The set S consists of all the positive integers that leave a remainder of 1 upon division by 4. The set T consists of all the positive integers that leave a remainder of 3 upon division by 4.

(i) Describe in words the sets $S \cup T$ and $S \cap T$.

(ii) Prove that the product of any two integers in S is also in S. Determine whether the product of any two integers in T is also in T.

(iii) Given an integer in T that is not a prime number, prove that at least one of its prime factors is in T.

(iv) For any set X of positive integers, an integer in X (other than 1) is said to be X-*prime* if it cannot be expressed as the product of two or more integers *all in* X (and all different from 1).

(a) Show that every integer in T is either T-prime or is the product of an odd number of T-prime integers.

(b) Find an example of an integer in S that can be expressed as the product of S-prime integers in two distinct ways. [Note: $s_1 s_2$ and $s_2 s_1$ are not counted as distinct ways of expressing the product of s_1 and s_2.]

8 Given an infinite sequence of numbers $u_0,\ u_1,\ u_2,\ \ldots$, we define the *generating function*, f, for the sequence by

$$\mathrm{f}(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots .$$

Issues of convergence can be ignored in this question.

(i) Using the binomial series, show that the sequence given by $u_n = n$ has generating function $x(1-x)^{-2}$, and find the sequence that has generating function $x(1-x)^{-3}$.

Hence, or otherwise, find the generating function for the sequence $u_n = n^2$. You should simplify your answer.

(ii) **(a)** The sequence $u_0,\ u_1,\ u_2,\ \ldots$ is determined by $u_n = ku_{n-1}$ $(n \geqslant 1)$, where k is independent of n, and $u_0 = a$. By summing the identity $u_n x^n \equiv ku_{n-1}x^n$, or otherwise, show that the generating function, f, satisfies

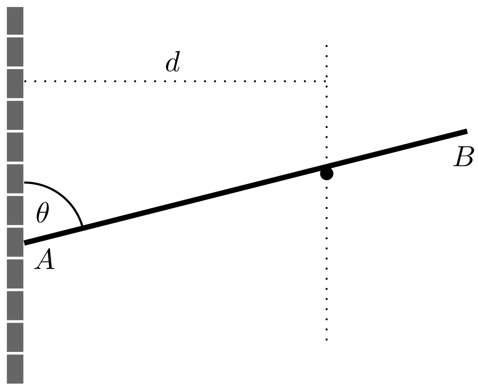
$$\mathrm{f}(x) = a + kx\mathrm{f}(x)\,.$$

Write down an expression for $\mathrm{f}(x)$.

(b) The sequence $u_0,\ u_1,\ u_2,\ \ldots$ is determined by $u_n = u_{n-1} + u_{n-2}$ $(n \geqslant 2)$ and $u_0 = 0$, $u_1 = 1$. Obtain the generating function.

Section B: Mechanics

9 A horizontal rail is fixed parallel to a vertical wall and at a distance d from the wall. A uniform rod AB of length $2a$ rests in equilibrium on the rail with the end A in contact with the wall. The rod lies in a vertical plane perpendicular to the wall. It is inclined at an angle θ to the vertical (where $0 < \theta < \frac{1}{2}\pi$) and $a\sin\theta < d$, as shown in the diagram.



The coefficient of friction between the rod and the wall is μ, and the coefficient of friction between the rod and the rail is λ.

Show that in limiting equilibrium, with the rod on the point of slipping at both the wall and the rail, the angle θ satisfies

$$d\,\mathrm{cosec}^2\,\theta = a\big((\lambda+\mu)\cos\theta + (1-\lambda\mu)\sin\theta\big)\,.$$

Derive the corresponding result if, instead, $a\sin\theta > d$.

10 Four particles A, B, C and D are initially at rest on a smooth horizontal table. They lie equally spaced a small distance apart, in the order $ABCD$, in a straight line. Their masses are λm, m, m and m, respectively, where $\lambda > 1$.

Particles A and D are simultaneously projected, both at speed u, so that they collide with B and C (respectively). In the following collision between B and C, particle B is brought to rest. The coefficient of restitution in each collision is e.

(i) Show that $e = \dfrac{\lambda - 1}{3\lambda + 1}$ and deduce that $e < \frac{1}{3}$.

(ii) Given also that C and D move towards each other with the same speed, find the value of λ and of e.

11 The point O is at the top of a vertical tower of height h which stands in the middle of a large horizontal plain. A projectile P is fired from O at a fixed speed u and at an angle α above the horizontal.

Show that the distance x from the base of the tower when P hits the plain satisfies

$$\frac{gx^2}{u^2} = h(1 + \cos 2\alpha) + x \sin 2\alpha\,.$$

Show that the greatest value of x as α varies occurs when $x = h \tan 2\alpha$ and find the corresponding value of $\cos 2\alpha$ in terms of g, h and u.

Show further that the greatest achievable distance between O and the landing point is $\dfrac{u^2}{g} + h\,.$

Section C: Probability and Statistics

12 **(i)** Alice tosses a fair coin twice and Bob tosses a fair coin three times. Calculate the probability that Bob gets more heads than Alice.

(ii) Alice tosses a fair coin three times and Bob tosses a fair coin four times. Calculate the probability that Bob gets more heads than Alice.

(iii) Let p_1 be the probability that Bob gets the same number of heads as Alice, and let p_2 be the probability that Bob gets more heads than Alice, when Alice and Bob each toss a fair coin n times.

Alice tosses a fair coin n times and Bob tosses a fair coin $n+1$ times. Express the probability that Bob gets more heads than Alice in terms of p_1 and p_2, and hence obtain a generalisation of the results of parts (i) and (ii).

13 An internet tester sends n e-mails simultaneously at time $t = 0$. Their arrival times at their destinations are independent random variables each having probability density function $\lambda e^{-\lambda t}$ $(0 \leqslant t < \infty,\ \lambda > 0)$.

(i) The random variable T is the time of arrival of the e-mail that arrives first at its destination. Show that the probability density function of T is

$$n\lambda e^{-n\lambda t}\,,$$

and find the expected value of T.

(ii) Write down the probability that the second e-mail to arrive at its destination arrives later than time t and hence derive the density function for the time of arrival of the second e-mail. Show that the expected time of arrival of the second e-mail is

$$\frac{1}{\lambda}\left(\frac{1}{n-1}+\frac{1}{n}\right).$$