Section A: Pure Mathematics

1 The curve C_1 has parametric equations $x = t^2$, $y = t^3$, where $-\infty < t < \infty$. Let O denote the point $(0, 0)$. The points P and Q on C_1 are such that $\angle POQ$ is a right angle. Show that the tangents to C_1 at P and Q intersect on the curve C_2 with equation $4y^2 = 3x - 1$.

Determine whether C_1 and C_2 meet, and sketch the two curves on the same axes.

2 Use the factor theorem to show that $a + b - c$ is a factor of

$$(a + b + c)^3 - 6(a + b + c)(a^2 + b^2 + c^2) + 8(a^3 + b^3 + c^3)\,. \qquad (*)$$

Hence factorise $(*)$ completely.

(i) Use the result above to solve the equation

$$(x + 1)^3 - 3(x + 1)(2x^2 + 5) + 2(4x^3 + 13) = 0\,.$$

(ii) By setting $d + e = c$, or otherwise, show that $(a + b - d - e)$ is a factor of

$$(a + b + d + e)^3 - 6(a + b + d + e)(a^2 + b^2 + d^2 + e^2) + 8(a^3 + b^3 + d^3 + e^3)$$

and factorise this expression completely.

Hence solve the equation

$$(x + 6)^3 - 6(x + 6)(x^2 + 14) + 8(x^3 + 36) = 0\,.$$

3 For each non-negative integer n, the polynomial f_n is defined by

$$\mathrm{f}_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\,.$$

(i) Show that $\mathrm{f}'_n(x) = \mathrm{f}_{n-1}(x)$ (for $n \geqslant 1$).

(ii) Show that, if a is a real root of the equation

$$\mathrm{f}_n(x) = 0\,, \qquad (*)$$

then $a < 0$.

(iii) Let a and b be distinct real roots of $(*)$, for $n \geqslant 2$. Show that $\mathrm{f}'_n(a)\,\mathrm{f}'_n(b) > 0$ and use a sketch to deduce that $\mathrm{f}_n(c) = 0$ for some number c between a and b.

Deduce that $(*\ )$ has at most one real root. How many real roots does $(*)$ have if n is odd? How many real roots does $(*)$ have if n is even?

4 Let

$$y = \frac{x^2 + x\sin\theta + 1}{x^2 + x\cos\theta + 1}\,.$$

(i) Given that x is real, show that

$$(y\cos\theta - \sin\theta)^2 \geqslant 4(y-1)^2\,.$$

Deduce that

$$y^2 + 1 \geqslant 4(y-1)^2\,,$$

and hence that

$$\frac{4-\sqrt{7}}{3} \leqslant y \leqslant \frac{4+\sqrt{7}}{3}\,.$$

(ii) In the case $y = \dfrac{4+\sqrt{7}}{3}$, show that

$$\sqrt{y^2+1} = 2(y-1)$$

and find the corresponding values of x and $\tan\theta$.

5 In this question, the definition of $\binom{p}{q}$ is taken to be

$$\binom{p}{q} = \begin{cases} \dfrac{p!}{q!(p-q)!} & \text{if } p \geqslant q \geqslant 0\,, \\ 0 & \text{otherwise}\ . \end{cases}$$

(i) Write down the coefficient of x^n in the binomial expansion for $(1-x)^{-N}$, where N is a positive integer, and write down the expansion using the Σ summation notation.

By considering $(1-x)^{-1}(1-x)^{-N}$, where N is a positive integer, show that

$$\sum_{j=0}^{n} \binom{N+j-1}{j} = \binom{N+n}{n}.$$

(ii) Show that, for any positive integers m, n and r with $r \leqslant m+n$,

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{j}\binom{n}{r-j}.$$

(iii) Show that, for any positive integers m and N,

$$\sum_{j=0}^{n} (-1)^j \binom{N+m}{n-j}\binom{m+j-1}{j} = \binom{N}{n}.$$

6 This question concerns solutions of the differential equation

$$(1-x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + k^2y^2 = k^2 \qquad (*)$$

where k is a positive integer.

For each value of k, let $y_k(x)$ be the solution of $(*)$ that satisfies $y_k(1) = 1$; you may assume that there is only one such solution for each value of k.

(i) Write down the differential equation satisfied by $y_1(x)$ and verify that $y_1(x) = x$.

(ii) Write down the differential equation satisfied by $y_2(x)$ and verify that $y_2(x) = 2x^2 - 1$.

(iii) Let $z(x) = 2\big(y_n(x)\big)^2 - 1$. Show that

$$(1-x^2)\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + 4n^2z^2 = 4n^2$$

and hence obtain an expression for $y_{2n}(x)$ in terms of $y_n(x)$.

(iv) Let $v(x) = y_n\big(y_m(x)\big)$. Show that $v(x) = y_{mn}(x)$.

7 Show that

$$\int_0^a \mathrm{f}(x)\mathrm{d}x = \int_0^a \mathrm{f}(a-x)\mathrm{d}x\,, \qquad (*)$$

where f is any function for which the integrals exist.

(i) Use $(*)$ to evaluate

$$\int_0^{\frac{1}{2}\pi} \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x\,.$$

(ii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x\,.$$

(iii) Evaluate

$$\int_0^{\frac{1}{4}\pi} \ln(1+\tan x)\,\mathrm{d}x\,.$$

(iv) Evaluate

$$\int_0^{\frac{1}{4}\pi} \frac{x}{\cos x\,(\cos x + \sin x)}\,\mathrm{d}x\,.$$

8 Evaluate the integral

$$\int_{m-\frac{1}{2}}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \qquad (m > \tfrac{1}{2})\,.$$

Show by means of a sketch that

$$\sum_{r=m}^{n} \frac{1}{r^2} \approx \int_{m-\frac{1}{2}}^{n+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x\,, \qquad (*)$$

where m and n are positive integers with $m < n$.

(i) You are given that the infinite series $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2}$ converges to a value denoted by E. Use $(*)$ to obtain the following approximations for E:

$$E \approx 2\,; \qquad E \approx \frac{5}{3}\,; \qquad E \approx \frac{33}{20}\,.$$

(ii) Show that, when r is large, the error in approximating $\dfrac{1}{r^2}$ by $\displaystyle\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} \frac{1}{x^2}\,\mathrm{d}x$ is approximately $\dfrac{1}{4r^4}$.

Given that $E \approx 1.645$, show that $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^4} \approx 1.08\,.$

Section B: Mechanics

9 A small bullet of mass m is fired into a block of wood of mass M which is at rest. The speed of the bullet on entering the block is u. Its trajectory within the block is a horizontal straight line and the resistance to the bullet's motion is R, which is constant.

(i) The block is fixed. The bullet travels a distance a inside the block before coming to rest. Find an expression for a in terms of m, u and R.

(ii) Instead, the block is free to move on a smooth horizontal table. The bullet travels a distance b inside the block before coming to rest relative to the block, at which time the block has moved a distance c on the table. Find expressions for b and c in terms of M, m and a.

10 A thin uniform wire is bent into the shape of an isosceles triangle ABC, where AB and AC are of equal length and the angle at A is 2θ. The triangle ABC hangs on a small rough horizontal peg with the side BC resting on the peg. The coefficient of friction between the wire and the peg is μ. The plane containing ABC is vertical. Show that the triangle can rest in equilibrium with the peg in contact with any point on BC provided

$$\mu \geqslant 2\tan\theta(1+\sin\theta)\,.$$

11 **(i)** Two particles move on a smooth horizontal surface. The positions, in Cartesian coordinates, of the particles at time t are $(a+ut\cos\alpha\,,\ ut\sin\alpha)$ and $(vt\cos\beta\,,\ b+vt\sin\beta)$, where a, b, u and v are positive constants, α and β are constant acute angles, and $t \geqslant 0$.

Given that the two particles collide, show that

$$u\sin(\theta+\alpha) = v\sin(\theta+\beta)\,,$$

where θ is the acute angle satisfying $\tan\theta = \dfrac{b}{a}$.

(ii) A gun is placed on the top of a vertical tower of height b which stands on horizontal ground. The gun fires a bullet with speed v and (acute) angle of elevation β. Simultaneously, a target is projected from a point on the ground a horizontal distance a from the foot of the tower. The target is projected with speed u and (acute) angle of elevation α, in a direction directly away from the tower.

Given that the target is hit before it reaches the ground, show that

$$2u\sin\alpha(u\sin\alpha - v\sin\beta) > bg\,.$$

Explain, with reference to part (i), why the target can only be hit if $\alpha > \beta$.

Section C: Probability and Statistics

12 Starting with the result $\mathrm{P}(A \cup B) = \mathrm{P}(A) + P(B) - \mathrm{P}(A \cap B)$, prove that

$$\mathrm{P}(A \cup B \cup C) = \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(B \cap C) - \mathrm{P}(C \cap A) + \mathrm{P}(A \cap B \cap C)\,.$$

Write down, without proof, the corresponding result for four events A, B, C and D.

A pack of n cards, numbered 1, 2, ..., n, is shuffled and laid out in a row. The result of the shuffle is that each card is equally likely to be in any position in the row. Let E_i be the event that the card bearing the number i is in the ith position in the row. Write down the following probabilities:

(i) $\mathrm{P}(E_i)$;

(ii) $\mathrm{P}(E_i \cap E_j)$, where $i \neq j$;

(iii) $\mathrm{P}(E_i \cap E_j \cap E_k)$, where $i \neq j$, $j \neq k$ and $k \neq i$.

Hence show that the probability that at least one card is in the same position as the number it bears is

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + (-1)^{n+1}\frac{1}{n!}\,.$$

Find the probability that exactly one card is in the same position as the number it bears.

13 **(i)** The random variable X has a binomial distribution with parameters n and p, where $n = 16$ and $p = \frac{1}{2}$. Show, using an approximation in terms of the standard normal density function $\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}x^2}$, that

$$\mathrm{P}(X = 8) \approx \frac{1}{2\sqrt{2\pi}}\,.$$

(ii) By considering a binomial distribution with parameters $2n$ and $\frac{1}{2}$, show that

$$(2n)! \approx \frac{2^{2n}(n!)^2}{\sqrt{n\pi}}\,.$$

(iii) By considering a Poisson distribution with parameter n, show that

$$n! \approx \sqrt{2\pi n}\,\mathrm{e}^{-n}\,n^n\,.$$