Section A: Pure Mathematics

1 Let

$$I_n = \int_{-\infty}^{\infty} \frac{1}{(x^2 + 2ax + b)^n} \, \mathrm{d}x \,,$$

where a and b are constants with $b > a^2$, and n is a positive integer.

(i) By using the substitution $x + a = \sqrt{b - a^2} \tan u$, or otherwise, show that

$$I_1 = \frac{\pi}{\sqrt{b - a^2}} \,.$$

(ii) Show that $2n(b - a^2)\, I_{n+1} = (2n - 1)\, I_n$.

(iii) Hence prove by induction that

$$I_n = \frac{\pi}{2^{2n-2}(b - a^2)^{n - \frac{1}{2}}} \binom{2n-2}{n-1} \,.$$

2 The distinct points $P(ap^2, 2ap)$, $Q(aq^2, 2aq)$ and $R(ar^2, 2ar)$ lie on the parabola $y^2 = 4ax$, where $a > 0$. The points are such that the normal to the parabola at Q and the normal to the parabola at R both pass through P.

(i) Show that $q^2 + qp + 2 = 0$.

(ii) Show that QR passes through a certain point that is independent of the choice of P.

(iii) Let T be the point of intersection of OP and QR, where O is the coordinate origin. Show that T lies on a line that is independent of the choice of P.

Show further that the distance from the x-axis to T is less than $\dfrac{a}{\sqrt{2}}$.

3 **(i)** Given that

$$\int \frac{x^3 - 2}{(x+1)^2}\, \mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\, \mathrm{e}^x + \text{constant}\,,$$

where $\mathrm{P}(x)$and $\mathrm{Q}(x)$ are polynomials, show that $\mathrm{Q}(x)$ has a factor of $x + 1$.

Show also that the degree of $\mathrm{P}(x)$ is exactly one more than the degree of $\mathrm{Q}(x)$, and find $\mathrm{P}(x)$ in the case $\mathrm{Q}(x) = x + 1$.

(ii) Show that there are no polynomials $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ such that

$$\int \frac{1}{x+1}\, \mathrm{e}^x \mathrm{d}x = \frac{\mathrm{P}(x)}{\mathrm{Q}(x)}\, \mathrm{e}^x + \text{constant}\,.$$

You need consider only the case when $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ have no common factors.

4 **(i)** By considering $\dfrac{1}{1+x^r} - \dfrac{1}{1+x^{r+1}}$ for $|x| \neq 1$, simplify

$$\sum_{r=1}^{N} \frac{x^r}{(1+x^r)(1+x^{r+1})}\,.$$

Show that, for $|x| < 1$,

$$\sum_{r=1}^{\infty} \frac{x^r}{(1+x^r)(1+x^{r+1})} = \frac{x}{1-x^2}\,.$$

(ii) Deduce that

$$\sum_{r=1}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y) = 2\mathrm{e}^{-y}\operatorname{cosech}(2y)$$

for $y > 0$.

Hence simplify

$$\sum_{r=-\infty}^{\infty} \operatorname{sech}(ry)\operatorname{sech}((r+1)y)\,,$$

for $y > 0$.

5 **(i)** By considering the binomial expansion of $(1+x)^{2m+1}$, prove that

$$\binom{2m+1}{m} < 2^{2m},$$

for any positive integer m.

(ii) For any positive integers r and s with $r < s$, $P_{r,s}$ is defined as follows: $P_{r,s}$ is the product of all the prime numbers greater than r and less than or equal to s, if there are any such primes numbers; if there are no such primes numbers, then $P_{r,s} = 1$.

For example, $P_{3,7} = 35$, $P_{7,10} = 1$ and $P_{14,18} = 17$.

Show that, for any positive integer m, $P_{m+1\,,\,2m+1}$ divides $\binom{2m+1}{m}$, and deduce that

$$P_{m+1\,,\,2m+1} < 2^{2m}.$$

(iii) Show that, if $P_{1,k} < 4^k$ for $k = 2, 3, \ldots, 2m$, then $P_{1,2m+1} < 4^{2m+1}$.

(iv) Prove that $\mathrm{P}_{1,n} < 4^n$ for $n \geqslant 2$.

6 Show, by finding R and γ, that $A\sinh x + B\cosh x$ can be written in the form $R\cosh(x+\gamma)$ if $B > A > 0$. Determine the corresponding forms in the other cases that arise, for $A > 0$, according to the value of B.

Two curves have equations $y = \operatorname{sech} x$ and $y = a\tanh x + b$, where $a > 0$.

(i) In the case $b > a$, show that if the curves intersect then the x-coordinates of the points of intersection can be written in the form

$$\pm\operatorname{arcosh}\left(\frac{1}{\sqrt{b^2-a^2}}\right) - \operatorname{artanh}\frac{a}{b}.$$

(ii) Find the corresponding result in the case $a > b > 0$.

(iii) Find necessary and sufficient conditions on a and b for the curves to intersect at two distinct points.

(iv) Find necessary and sufficient conditions on a and b for the curves to touch and, given that they touch, express the y-coordinate of the point of contact in terms of a.

7 Let $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$, where n is a positive integer. Show that, for any complex number z,

$$(z-1)(z-\omega)\cdots(z-\omega^{n-1}) = z^n - 1\,.$$

The points X_0, X_1, ... , X_{n-1} lie on a circle with centre O and radius 1, and are the vertices of a regular polygon.

(i) The point P is equidistant from X_0 and X_1. Show that, if n is even,

$$|PX_0| \times |PX_1| \times \cdots \times |PX_{n-1}| = |OP|^n + 1\,,$$

where $|PX_k|$ denotes the distance from P to X_k.

Give the corresponding result when n is odd. (There are two cases to consider.)

(ii) Show that

$$|X_0X_1| \times |X_0X_2| \times \cdots \times |X_0X_{n-1}| = n\,.$$

8 **(i)** The function f satisfies, for all x, the equation

$$\mathrm{f}(x) + (1-x)\mathrm{f}(-x) = x^2\,.$$

Show that $\mathrm{f}(-x) + (1+x)\mathrm{f}(x) = x^2$. Hence find $\mathrm{f}(x)$ in terms of x. You should verify that your function satisfies the original equation.

(ii) The function K is defined, for $x \neq 1$, by

$$\mathrm{K}(x) = \frac{x+1}{x-1}\,.$$

Show that, for $x \neq 1$, $\mathrm{K}(\mathrm{K}(x)) = x$.

The function g satisfies the equation

$$\mathrm{g}(x) + x\,\mathrm{g}\Big(\frac{x+1}{x-1}\Big) = x \qquad (x \neq 1)\,.$$

Show that, for $x \neq 1$, $\mathrm{g}(x) = \dfrac{2x}{x^2+1}$.

(iii) Find $\mathrm{h}(x)$, for $x \neq 0$, $x \neq 1$, given that

$$\mathrm{h}(x) + \mathrm{h}\Big(\frac{1}{1-x}\Big) = 1 - x - \frac{1}{1-x} \qquad (x \neq 0,\quad x \neq 1)\,.$$

Section B: Mechanics

9 Three pegs P, Q and R are fixed on a smooth horizontal table in such a way that they form the vertices of an equilateral triangle of side $2a$. A particle X of mass m lies on the table. It is attached to the pegs by three springs, PX, QX and RX, each of modulus of elasticity λ and natural length l, where $l < \frac{2}{\sqrt{3}}\,a$. Initially the particle is in equilibrium. Show that the extension in each spring is $\frac{2}{\sqrt{3}}\,a - l$.

The particle is then pulled a small distance directly towards P and released. Show that the tension T in the spring RX is given by

$$T = \frac{\lambda}{l}\left(\sqrt{\frac{4a^2}{3} + \frac{2ax}{\sqrt{3}} + x^2}\; - l\right),$$

where x is the displacement of X from its equilibrium position.

Show further that the particle performs approximate simple harmonic motion with period

$$2\pi\sqrt{\frac{4mla}{3(4a - \sqrt{3}\,l)\lambda}}\;.$$

10 A smooth plane is inclined at an angle α to the horizontal. A particle P of mass m is attached to a fixed point A above the plane by a light inextensible string of length a. The particle rests in equilibrium on the plane, and the string makes an angle β with the plane.

The particle is given a horizontal impulse parallel to the plane so that it has an initial speed of u. Show that the particle will not immediately leave the plane if $ag\cos(\alpha+\beta) > u^2\tan\beta$.

Show further that a necessary condition for the particle to perform a complete circle whilst in contact with the plane is $6\tan\alpha\tan\beta < 1$.

11 A car of mass m travels along a straight horizontal road with its engine working at a constant rate P. The resistance to its motion is such that the acceleration of the car is zero when it is moving with speed $4U$.

(i) Given that the resistance is proportional to the car's speed, show that the distance X_1 travelled by the car while it accelerates from speed U to speed $2U$, is given by

$$\lambda X_1 = 2\ln\tfrac{9}{5} - 1\,,$$

where $\lambda = P/(16mU^3)$.

(ii) Given instead that the resistance is proportional to the square of the car's speed, show that the distance X_2 travelled by the car while it accelerates from speed U to speed $2U$ is given by

$$\lambda X_2 = \tfrac{4}{3}\ln\tfrac{9}{8}\,.$$

(iii) Given that $3.17 < \ln 24 < 3.18$ and $1.60 < \ln 5 < 1.61$, determine which is the larger of X_1 and X_2.

Section C: Probability and Statistics

12 Let X be a random variable with mean μ and standard deviation σ. *Chebyshev's inequality*, which you may use without proof, is

$$\mathrm{P}\left(|X-\mu| > k\sigma\right) \leqslant \frac{1}{k^2}\,,$$

where k is any positive number.

(i) The probability of a biased coin landing heads up is 0.2. It is thrown $100n$ times, where n is an integer greater than 1. Let α be the probability that the coin lands heads up N times, where $16n \leqslant N \leqslant 24n$.

Use Chebyshev's inequality to show that

$$\alpha \geqslant 1 - \frac{1}{n}\,.$$

(ii) Use Chebyshev's inequality to show that

$$1 + n + \frac{n^2}{2!} + \cdots + \frac{n^{2n}}{(2n)!} \geqslant \left(1 - \frac{1}{n}\right)\mathrm{e}^n\,.$$

13 Given a random variable X with mean μ and standard deviation σ, we define the *kurtosis*, κ, of X by

$$\kappa = \frac{\mathrm{E}\big((X-\mu)^4\big)}{\sigma^4} - 3\,.$$

Show that the random variable $X - a$, where a is a constant, has the same kurtosis as X.

(i) Show by integration that a random variable which is Normally distributed with mean 0 has kurtosis 0.

(ii) Let $Y_1, Y_2, \ldots, Y_n$ be n independent, identically distributed, random variables with mean 0, and let $T = \sum\limits_{r=1}^{n} Y_r$. Show that

$$\mathrm{E}(T^4) = \sum_{r=1}^{n} \mathrm{E}(Y_r^4) + 6\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} \mathrm{E}(Y_s^2)\mathrm{E}(Y_r^2)\,.$$

(iii) Let $X_1, X_2, \ldots, X_n$ be n independent, identically distributed, random variables each with kurtosis κ. Show that the kurtosis of their sum is $\dfrac{\kappa}{n}$.