Section A: Pure Mathematics

1 **(i)** Use the substitution $u = x\sin x + \cos x$ to find

$$\int \frac{x}{x\tan x + 1}\,\mathrm{d}x\,.$$

Find by means of a similar substitution, or otherwise,

$$\int \frac{x}{x\cot x - 1}\,\mathrm{d}x\,.$$

(ii) Use a substitution to find

$$\int \frac{x\sec^2 x\,\tan x}{x\sec^2 x - \tan x}\,\mathrm{d}x$$

and

$$\int \frac{x\sin x\cos x}{(x - \sin x\cos x)^2}\,\mathrm{d}x\,.$$

2 **(i)** The inequality $\dfrac{1}{t} \leqslant 1$ holds for $t \geqslant 1$. By integrating both sides of this inequality over the interval $1 \leqslant t \leqslant x$, show that

$$\ln x \leqslant x - 1 \qquad (*)$$

for $x \geqslant 1$. Show similarly that $(*)$ also holds for $0 < x \leqslant 1$.

(ii) Starting from the inequality $\dfrac{1}{t^2} \leqslant \dfrac{1}{t}$ for $t \geqslant 1$, show that

$$\ln x \geqslant 1 - \frac{1}{x} \qquad (**)$$

for $x > 0$.

(iii) Show, by integrating $(*)$ and $(**)$, that

$$\frac{2}{y+1} \leqslant \frac{\ln y}{y-1} \leqslant \frac{y+1}{2y}$$

for $y > 0$ and $y \neq 1$.

3 The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$, where $p > 0$ and $q < 0$, lie on the curve C with equation

$$y^2 = 4ax\,,$$

where $a > 0$. Show that the equation of the tangent to C at P is

$$y = \frac{1}{p}\,x + ap\,.$$

The tangents to the curve at P and at Q meet at R. These tangents meet the y-axis at S and T respectively, and O is the origin. Prove that the area of triangle OPQ is twice the area of triangle RST.

4 **(i)** Let r be a real number with $|r| < 1$ and let

$$S = \sum_{n=0}^{\infty} r^n\,.$$

You may assume without proof that $S = \dfrac{1}{1-r}$.

Let $p = 1 + r + r^2$. Sketch the graph of the function $1 + r + r^2$ and deduce that $\frac{3}{4} \leqslant p < 3$.

Show that, if $1 < p < 3$, then the value of p determines r, and hence S, uniquely.

Show also that, if $\frac{3}{4} < p < 1$, then there are two possible values of S and these values satisfy the equation $(3-p)S^2 - 3S + 1 = 0$.

(ii) Let r be a real number with $|r| < 1$ and let

$$T = \sum_{n=1}^{\infty} nr^{n-1}\,.$$

You may assume without proof that $T = \dfrac{1}{(1-r)^2}$.

Let $q = 1 + 2r + 3r^2$. Find the set of values of q that determine T uniquely.

Find the set of values of q for which T has two possible values. Find also a quadratic equation, with coefficients depending on q, that is satisfied by these two values.

5 A circle of radius a is centred at the origin O. A rectangle $PQRS$ lies in the minor sector OMN of this circle where M is $(a, 0)$ and N is $(a\cos\beta, a\sin\beta)$, and β is a constant with $0 < \beta < \frac{\pi}{2}$. Vertex P lies on the positive x-axis at $(x, 0)$; vertex Q lies on ON; vertex R lies on the arc of the circle between M and N; and vertex S lies on the positive x-axis at $(s, 0)$.

Show that the area A of the rectangle can be written in the form

$$A = x(s - x)\tan\beta\,.$$

Obtain an expression for s in terms of a, x and β, and use it to show that

$$\frac{\mathrm{d}A}{\mathrm{d}x} = (s - 2x)\tan\beta - \frac{x^2}{s}\tan^3\beta\,.$$

Deduce that the greatest possible area of rectangle $PQRS$ occurs when $s = x(1 + \sec\beta)$ and show that this greatest area is $\frac{1}{2}a^2\tan\frac{1}{2}\beta$.

Show also that this greatest area occurs when $\angle ROS = \frac{1}{2}\beta$.

6 In this question, you may assume that, if a continuous function takes both positive and negative values in an interval, then it takes the value 0 at some point in that interval.

(i) The function f is continuous and $\mathrm{f}(x)$ is non-zero for some value of x in the interval $0 \leqslant x \leqslant 1$. Prove by contradiction, or otherwise, that if

$$\int_0^1 \mathrm{f}(x)\mathrm{d}x = 0\,,$$

then $\mathrm{f}(x)$ takes both positive and negative values in the interval $0 \leqslant x \leqslant 1$.

(ii) The function g is continuous and

$$\int_0^1 \mathrm{g}(x)\,\mathrm{d}x = 1\,, \quad \int_0^1 x\mathrm{g}(x)\,\mathrm{d}x = \alpha\,, \quad \int_0^1 x^2\mathrm{g}(x)\,\mathrm{d}x = \alpha^2\,. \qquad (*)$$

Show, by considering

$$\int_0^1 (x - \alpha)^2\mathrm{g}(x)\,\mathrm{d}x\,,$$

that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

Find a function of the form $\mathrm{g}(x) = a + bx$ that satisfies the conditions $(*)$ and verify that $\mathrm{g}(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

(iii) The function h has a continuous derivative h′ and

$$\mathrm{h}(0) = 0\,, \quad \mathrm{h}(1) = 1\,, \quad \int_0^1 \mathrm{h}(x)\,\mathrm{d}x = \beta\,, \quad \int_0^1 x\mathrm{h}(x)\,\mathrm{d}x = \tfrac{1}{2}\beta(2 - \beta)\,.$$

Use the result in part **(ii)** to show that $\mathrm{h}'(x) = 0$ for some value of x in the interval $0 \leqslant x \leqslant 1$.

7 The triangle ABC has side lengths $|BC| = a$, $|CA| = b$ and $|AB| = c$. Equilateral triangles BXC, CYA and AZB are erected on the sides of the triangle ABC, with X on the other side of BC from A, and similarly for Y and Z. Points L, M and N are the centres of rotational symmetry of triangles BXC, CYA and AZB respectively.

(i) Show that $|CM| = \dfrac{b}{\sqrt{3}}$ and write down the corresponding expression for $|CL|$.

(ii) Use the cosine rule to show that

$$6\,|LM|^2 = a^2 + b^2 + c^2 + 4\sqrt{3}\,\Delta\,,$$

where Δ is the area of triangle ABC. Deduce that LMN is an equilateral triangle.

Show further that the areas of triangles LMN and ABC are equal if and only if

$$a^2 + b^2 + c^2 = 4\sqrt{3}\,\Delta\,.$$

(iii) Show that the conditions

$$(a-b)^2 = -2ab\big(1 - \cos(C - 60^\circ)\big)$$

and

$$a^2 + b^2 + c^2 = 4\sqrt{3}\,\Delta$$

are equivalent.

Deduce that the areas of triangles LMN and ABC are equal if and only if ABC is equilateral.

8 Two sequences are defined by $a_1 = 1$ and $b_1 = 2$ and, for $n \geqslant 1$,

$$\begin{aligned} a_{n+1} &= a_n + 2b_n\,, \\ b_{n+1} &= 2a_n + 5b_n\,. \end{aligned}$$

Prove by induction that, for all $n \geqslant 1$,

$$a_n^2 + 2a_nb_n - b_n^2 = 1\,. \qquad (*)$$

(i) Let $c_n = \dfrac{a_n}{b_n}$. Show that $b_n \geqslant 2 \times 5^{n-1}$ and use $(*)$ to show that

$$c_n \to \sqrt{2} - 1 \text{ as } n \to \infty\,.$$

(ii) Show also that $c_n > \sqrt{2} - 1$ and hence that $\dfrac{2}{c_n + 1} < \sqrt{2} < c_n + 1\,.$

Deduce that $\dfrac{140}{99} < \sqrt{2} < \dfrac{99}{70}\,.$

Section B: Mechanics

9 A particle is projected at speed u from a point O on a horizontal plane. It passes through a fixed point P which is at a horizontal distance d from O and at a height $d\tan\beta$ above the plane, where $d > 0$ and β is an acute angle. The angle of projection α is chosen so that u is as small as possible.

(i) Show that $u^2 = gd\tan\alpha$ and $2\alpha = \beta + 90°$.

(ii) At what angle to the horizontal is the particle travelling when it passes through P? Express your answer in terms of α in its simplest form.

10 Particles P_1, P_2, ... are at rest on the x-axis, and the x-coordinate of P_n is n. The mass of P_n is $\lambda^n m$. Particle P, of mass m, is projected from the origin at speed u towards P_1. A series of collisions takes place, and the coefficient of restitution at each collision is e, where $0 < e < 1$. The speed of P_n immediately after its first collision is u_n and the speed of P_n immediately after its second collision is v_n. No external forces act on the particles.

(i) Show that $u_1 = \dfrac{1+e}{1+\lambda}u$ and find expressions for u_n and v_n in terms of e, λ, u and n.

(ii) Show that, if $e > \lambda$, then each particle (except P) is involved in exactly two collisions.

(iii) Describe what happens if $e = \lambda$ and show that, in this case, the fraction of the initial kinetic energy lost approaches e as the number of collisions increases.

(iv) Describe what happens if $\lambda e = 1$. What fraction of the initial kinetic energy is eventually lost in this case?

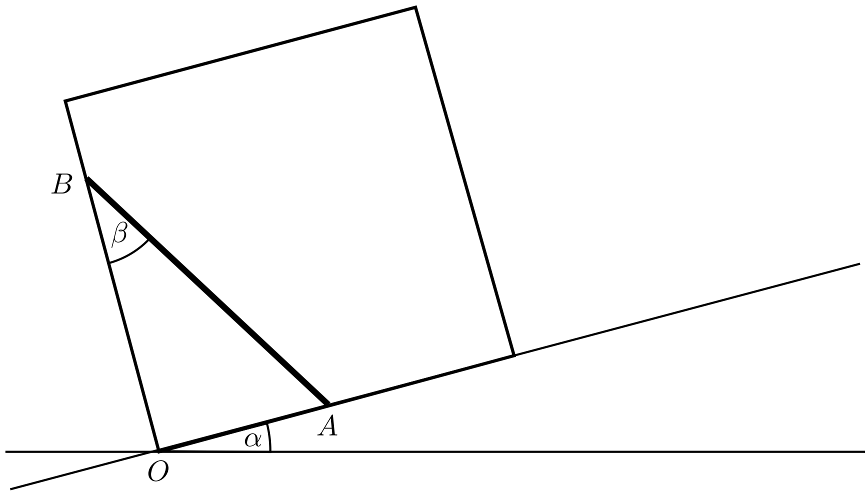
11 A plane makes an acute angle α with the horizontal. A box in the shape of a cube is fixed onto the plane in such a way that four of its edges are horizontal and two of its sides are vertical.

A uniform rod of length $2L$ and weight W rests with its lower end at A on the bottom of the box and its upper end at B on a side of the box, as shown in the diagram below. The vertical plane containing the rod is parallel to the vertical sides of the box and cuts the lowest edge of the box at O. The rod makes an acute angle β with the side of the box at B.

The coefficients of friction between the rod and the box at the two points of contact are both $\tan\gamma$, where $0 < \gamma < \frac{1}{2}\pi$.

The rod is in limiting equilibrium, with the end at A on the point of slipping in the direction away from O and the end at B on the point of slipping towards O. Given that $\alpha < \beta$, show that $\beta = \alpha + 2\gamma$.

[**Hint**: You may find it helpful to take moments about the midpoint of the rod.]



Section C: Probability and Statistics

12 In a lottery, each of the N participants pays £c to the organiser and picks a number from 1 to N. The organiser picks at random the winning number from 1 to N and all those participants who picked this number receive an equal share of the prize, £J.

(i) The participants pick their numbers independently and with equal probability. Obtain an expression for the probability that no participant picks the winning number, and hence determine the organiser's expected profit.

Use the approximation

$$\left(1 - \frac{a}{N}\right)^N \approx \mathrm{e}^{-a} \qquad (*)$$

to show that if $2Nc = J$ then the organiser will expect to make a loss.

Note: $\mathrm{e} > 2$.

(ii) Instead of the numbers being equally popular, a fraction γ of the numbers are popular and the rest are unpopular. For each participant, the probability of picking any given popular number is $\dfrac{a}{N}$ and the probability of picking any given unpopular number is $\dfrac{b}{N}$.

Find a relationship between a, b and γ.

Show that, using the approximation $(*)$, the organiser's expected profit can be expressed in the form

$$A\mathrm{e}^{-a} + B\mathrm{e}^{-b} + C\,,$$

where A, B and C can be written in terms of J, c, N and γ.

In the case $\gamma = \frac{1}{8}$ and $a = 9b$, find a and b. Show that, if $2Nc = J$, then the organiser will expect to make a profit.

Note: $\mathrm{e} < 3$.

13 I have a sliced loaf which initially contains n slices of bread. Each time I finish setting a STEP question, I make myself a snack: either toast, using one slice of bread; or a sandwich, using two slices of bread. I make toast with probability p and I make a sandwich with probability q, where $p + q = 1$, unless there is only one slice left in which case I must, of course, make toast.

Let s_r $(1 \leqslant r \leqslant n)$ be the probability that the rth slice of bread is the second of two slices used to make a sandwich and let t_r $(1 \leqslant r \leqslant n)$ be the probability that the rth slice of bread is used to make toast. What is the value of s_1?

Explain why the following equations hold:

$$t_r = (s_{r-1} + t_{r-1})\,p \qquad (2 \leqslant r \leqslant n-1)\,;$$
$$s_r = 1 - (s_{r-1} + t_{r-1}) \qquad (2 \leqslant r \leqslant n)\,.$$

Hence, or otherwise, show that $s_r = q(1 - s_{r-1})$ for $2 \leqslant r \leqslant n - 1$.

Show further that

$$s_r = \frac{q + (-q)^r}{1+q} \qquad (1 \leqslant r \leqslant n-1)\,,$$

and find the corresponding expression for t_r.

Find also expressions for s_n and t_n in terms of q.