Section A: Pure Mathematics

1 **Note:** In this question you may use without proof the result $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(\arctan x\big) = \dfrac{1}{1+x^2}$.

Let

$$I_n = \int_0^1 x^n \arctan x \,\mathrm{d}x \ ,$$

where $n = 0,\ 1,\ 2,\ 3,\ \ldots$.

(i) Show that, for $n \geqslant 0$,

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x$$

and evaluate I_0.

(ii) Find an expression, in terms of n, for $(n+3)I_{n+2} + (n+1)I_n$.

Use this result to evaluate I_4.

(iii) Prove by induction that, for $n \geqslant 1$,

$$(4n+1)I_{4n} = A - \frac{1}{2}\sum_{r=1}^{2n}(-1)^r\frac{1}{r}\ ,$$

where A is a constant to be determined.

2 The sequence of numbers $x_0,\ x_1,\ x_2,\ \ldots$ satisfies

$$x_{n+1} = \frac{ax_n - 1}{x_n + b}\ .$$

(You may assume that a, b and x_0 are such that $x_n + b \neq 0$.)

Find an expression for x_{n+2} in terms of a, b and x_n.

(i) Show that $a + b = 0$ is a necessary condition for the sequence to be periodic with period 2.

Note: The sequence is said to be periodic with period k if $x_{n+k} = x_n$ for all n, and there is no integer m with $0 < m < k$ such that $x_{n+m} = x_n$ for all n.

(ii) Find necessary and sufficient conditions for the sequence to have period 4.

3 **(i)** Sketch, on x-y axes, the set of all points satisfying $\sin y = \sin x$, for $-\pi \leqslant x \leqslant \pi$ and $-\pi \leqslant y \leqslant \pi$. You should give the equations of all the lines on your sketch.

(ii) Given that

$$\sin y = \tfrac{1}{2}\sin x$$

obtain an expression, in terms of x, for y' when $0 \leqslant x \leqslant \frac{1}{2}\pi$ and $0 \leqslant y \leqslant \frac{1}{2}\pi$, and show that

$$y'' = -\frac{3\sin x}{(4-\sin^2 x)^{\frac{3}{2}}}\,.$$

Use these results to sketch the set of all points satisfying $\sin y = \frac{1}{2}\sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and $0 \leqslant y \leqslant \frac{1}{2}\pi$.

Hence sketch the set of all points satisfying $\sin y = \frac{1}{2}\sin x$ for $-\pi \leqslant x \leqslant \pi$ and $-\pi \leqslant y \leqslant \pi$.

(iii) Without further calculation, sketch the set of all points satisfying $\cos y = \frac{1}{2}\sin x$ for $-\pi \leqslant x \leqslant \pi$ and $-\pi \leqslant y \leqslant \pi$.

4 The Schwarz inequality is

$$\left(\int_a^b \mathrm{f}(x)\,\mathrm{g}(x)\,\mathrm{d}x\right)^2 \leqslant \left(\int_a^b \big(\mathrm{f}(x)\big)^2\mathrm{d}x\right)\left(\int_a^b \big(\mathrm{g}(x)\big)^2\mathrm{d}x\right). \qquad (*)$$

(i) By setting $\mathrm{f}(x) = 1$ in $(*)$, and choosing $\mathrm{g}(x)$, a and b suitably, show that for $t > 0$,

$$\frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \leqslant \frac{t}{2}\,.$$

(ii) By setting $\mathrm{f}(x) = x$ in $(*)$, and choosing $\mathrm{g}(x)$ suitably, show that

$$\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x \geqslant 12\big(1 - \mathrm{e}^{-\frac{1}{4}}\big)^2\,.$$

(iii) Use $(*)$ to show that

$$\frac{64}{25\pi} \leqslant \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\;\mathrm{d}x \leqslant \sqrt{\frac{\pi}{2}}\,.$$

5 A curve C is determined by the parametric equations

$$x = at^2\,,\ y = 2at\,,$$

where $a > 0$.

(i) Show that the normal to C at a point P, with non-zero parameter p, meets C again at a point N, with parameter n, where

$$n = -\left(p + \frac{2}{p}\right).$$

(ii) Show that the distance $|PN|$ is given by

$$|PN|^2 = 16a^2 \frac{(p^2+1)^3}{p^4}$$

and that this is minimised when $p^2 = 2$.

(iii) The point Q, with parameter q, is the point at which the circle with diameter PN cuts C again. By considering the gradients of QP and QN, show that

$$2 = p^2 - q^2 + \frac{2q}{p}\,.$$

Deduce that $|PN|$ is at its minimum when Q is at the origin.

6 Let

$$S_n = \sum_{r=1}^{n} \frac{1}{\sqrt{r}}\,,$$

where n is a positive integer.

(i) Prove by induction that

$$S_n \leqslant 2\sqrt{n} - 1\,.$$

(ii) Show that $(4k+1)\sqrt{k+1} > (4k+3)\sqrt{k}$ for $k \geqslant 0$.

Determine the smallest number C such that

$$S_n \geqslant 2\sqrt{n} + \frac{1}{2\sqrt{n}} - C\,.$$

7 The functions f and g are defined, for $x > 0$, by

$$\mathrm{f}(x) = x^x\,, \qquad \mathrm{g}(x) = x^{\mathrm{f}(x)}\,.$$

(i) By taking logarithms, or otherwise, show that $\mathrm{f}(x) > x$ for $0 < x < 1$. Show further that $x < \mathrm{g}(x) < \mathrm{f}(x)$ for $0 < x < 1$.

Write down the corresponding results for $x > 1$.

(ii) Find the value of x for which $\mathrm{f}'(x) = 0$.

(iii) Use the result $x \ln x \to 0$ as $x \to 0$ to find $\lim\limits_{x\to 0} \mathrm{f}(x)$, and write down $\lim\limits_{x\to 0} \mathrm{g}(x)$.

(iv) Show that $x^{-1} + \ln x \geqslant 1$ for $x > 0$.

Using this result, or otherwise, show that $\mathrm{g}'(x) > 0$.

Sketch the graphs, for $x > 0$, of $y = x$, $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$ on the same axes.

8 All vectors in this question lie in the same plane.

The vertices of the non-right-angled triangle ABC have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. The non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to BC and CA, respectively.

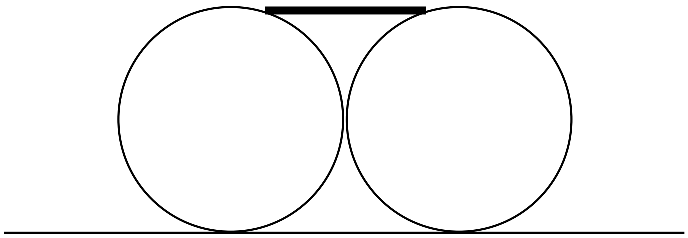
Write down the vector equation of the line through A perpendicular to BC, in terms of $\mathbf{u}$, $\mathbf{a}$ and a parameter λ.

The line through A perpendicular to BC intersects the line through B perpendicular to CA at P. Find the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{u}$.

Hence show that the line CP is perpendicular to the line AB.

Section B: Mechanics

9 Two identical rough cylinders of radius r and weight W rest, not touching each other but a negligible distance apart, on a horizontal floor. A thin flat rough plank of width $2a$, where $a < r$, and weight kW rests symmetrically and horizontally on the cylinders, with its length parallel to the axes of the cylinders and its faces horizontal. A vertical cross-section is shown in the diagram below.



The coefficient of friction at all four contacts is $\frac{1}{2}$. The system is in equilibrium.

(i) Let F be the frictional force between one cylinder and the floor, and let R be the normal reaction between the plank and one cylinder. Show that

$$R\sin\theta = F(1+\cos\theta)\,,$$

where θ is the acute angle between the plank and the tangent to the cylinder at the point of contact.

Deduce that $2\sin\theta \leqslant 1+\cos\theta\,$.

(ii) Show that

$$N = \left(1+\frac{2}{k}\right)\left(\frac{1+\cos\theta}{\sin\theta}\right)F\,,$$

where N is the normal reaction between the floor and one cylinder.

Write down the condition that the cylinder does not slip on the floor and show that it is satisfied with no extra restrictions on θ.

(iii) Show that $\sin\theta \leqslant \frac{4}{5}$ and hence that $r \leqslant 5a\,$.

10 A car of mass m makes a journey of distance $2d$ in a straight line. It experiences air resistance and rolling resistance so that the total resistance to motion when it is moving with speed v is $Av^2 + R$, where A and R are constants.

The car starts from rest and moves with constant acceleration a for a distance d. Show that the work done by the engine for this half of the journey is

$$\int_0^d (ma + R + Av^2)\,\mathrm{d}x$$

and that it can be written in the form

$$\int_0^w \frac{(ma + R + Av^2)v}{a}\,\mathrm{d}v\,,$$

where $w = \sqrt{2ad}$.

For the second half of the journey, the acceleration of the car is $-a$.

(i) In the case $R > ma$, show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd\,.$$

(ii) In the case $ma - 2Aad < R < ma$, show that at a certain speed the driving force required to maintain the constant acceleration falls to zero.

Thereafter, the engine does no work (and the driver applies the brakes to maintain the constant acceleration). Show that the work done by the engine for the whole journey is

$$2Aad^2 + 2Rd + \frac{(ma - R)^2}{4Aa}\,.$$

11 Two thin vertical parallel walls, each of height $2a$, stand a distance a apart on horizontal ground. The projectiles in this question move in a plane perpendicular to the walls.

(i) A particle is projected with speed $\sqrt{5ag}$ towards the two walls from a point A at ground level. It just clears the first wall. By considering the energy of the particle, find its speed when it passes over the first wall.

Given that it just clears the second wall, show that the angle its trajectory makes with the horizontal when it passes over the first wall is $45°$.

Find the distance of A from the foot of the first wall.

(ii) A second particle is projected with speed $\sqrt{5ag}$ from a point B at ground level towards the two walls. It passes a distance h above the first wall, where $h > 0$. Show that it does not clear the second wall.

Section C: Probability and Statistics

12 Adam and Eve are catching fish. The number of fish, X, that Adam catches in any time interval is Poisson distributed with parameter λt, where λ is a constant and t is the length of the time interval. The number of fish, Y, that Eve catches in any time interval is Poisson distributed with parameter μt, where μ is a constant and t is the length of the time interval

The two Poisson variables are independent. You may assume that that expected time between Adam catching a fish and Adam catching his next fish is λ^{-1}, and similarly for Eve.

(i) By considering $\mathrm{P}(X+Y=r)$, show that the total number of fish caught by Adam and Eve in time T also has a Poisson distribution.

(ii) Given that Adam and Eve catch a total of k fish in time T, where k is fixed, show that the number caught by Adam has a binomial distribution.

(iii) Given that Adam and Eve start fishing at the same time, find the probability that the first fish is caught by Adam.

(iv) Find the expected time from the moment Adam and Eve start fishing until they have each caught at least one fish.

[**Note** This question has been redrafted to make the meaning clearer.]

13 In a television game show, a contestant has to open a door using a key. The contestant is given a bag containing n keys, where $n \geqslant 2$. Only one key in the bag will open the door. There are three versions of the game. In each version, the contestant starts by choosing a key at random from the bag.

(i) In version 1, after each failed attempt at opening the door the key that has been tried is put back into the bag and the contestant again selects a key at random from the bag. By considering the binomial expansion of $(1-q)^{-2}$, or otherwise, find the expected number of attempts required to open the door.

(ii) In version 2, after each failed attempt at opening the door the key that has been tried is put aside and the contestant selects another key at random from the bag. Find the expected number of attempts required to open the door.

(iii) In version 3, after each failed attempt at opening the door the key that has been tried is put back into the bag and another incorrect key is added to the bag. The contestant then selects a key at random from the bag. Show that the probability that the contestant draws the correct key at the kth attempt is

$$\frac{n-1}{(n+k-1)(n+k-2)}\,.$$

Show also, using partial fractions, that the expected number of attempts required to open the door is infinite.

You may use without proof the result that $\displaystyle\sum_{m=1}^{N} \frac{1}{m} \to \infty$ as $N \to \infty$.